

Transformed Eulerian-Mean Theory. Part I: Nonquasigeostrophic Theory for Eddies on a Zonal-Mean Flow

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ABSTRACT

A theoretical formalism for nongeostrophic eddy transport in zonal-mean flows, using a transformed Eulerian-mean (TEM) approach in z coordinates, is discussed. By using Andrews and McIntyre's coordinate-independent definition of the "quasi-Stokes streamfunction," it is argued that the surface boundary condition can be dealt with more readily than when the widely used quasigeostrophic definition is adopted. Along with the "residual mean circulation," the concept of "residual eddy flux" arises naturally within the TEM framework, and it is argued that it is this residual eddy flux, and not the "raw" eddy flux, that might reasonably be expected to be downgradient. This distinction is shown to be especially important for Ertel potential vorticity (PV). The authors show how a closed set of transformed mean equations can be generated, and how the eddy forcing appears in the TEM momentum equations. Under adiabatic conditions, the "eddy drag" is just proportional to the residual eddy flux of PV along the mean isopycnals; in the diabatic layer close to the surface, it is more complicated, but becomes very simple for small Rossby number (without any assumption of small isopycnal slope).

1. Introduction

There has been much recent progress in understanding the role of mesoscale eddies in maintaining the thermal structure of the ocean (e.g., Karsten et al. 2002; Marshall et al. 2002; Radko and Marshall 2004) and in the theory and parameterization of eddy transport (e.g., Gent and McWilliams 1990; Gent et al. 1995; McDougall and McIntosh 1996; Treguier et al. 1997; Visbeck et al. 1997; Greatbatch 1998). Nevertheless, understanding of the role of eddies in the determination of ocean circulation and structure lags behind the corresponding atmospheric problem, especially for the stratosphere, where it is well known that eddy transports are almost entirely responsible for the observed departures of the zonal-mean state from radiative equilibrium, and the techniques for relating the degree of disequilibrium to the eddy fluxes of angular momentum are well established (e.g., Haynes et al. 1991). Part of the difficulty in the ocean case is that, except for simple models of the Antarctic Circumpolar Current (ACC), zonal averages are not appropriate and the considerable body of theory developed for understanding the zonal-mean problem is not directly applicable. Nevertheless (as we

hope to establish in what follows) there remain aspects of the dynamics of a zonally reentrant ocean that are worthy of further investigation and whose clarification is a useful stepping-stone to understanding more complex, and realistic, problems. In this paper, therefore, we focus on the behavior of eddy transport on a zonal-mean flow, and in Part II (Kuo et al. 2005) we shall apply these results to diagnosis of the equilibrium state of a wind- and buoyancy-driven flow in a cylinder. [A preliminary discussion of extension of this work to the three-dimensional case is presented in Ferrari and Plumb (2003).]

Theoretical formulations of eddy-mean-flow interaction under near-adiabatic conditions are most elegant, and most powerful, within an isentropic- or isopycnal-coordinate framework (e.g., Andrews 1983; Greatbatch 1998). However, working with isopycnal coordinates can be problematic near the surface where isopycnals outcrop in the presence of eddies. While this near-surface region is coming to be better understood, and there are ways of handling outcropping isopycnals (Held and Schneider 1999; Koh and Plumb 2004), here we prefer to use z coordinates.

To facilitate further analysis, therefore, we adopt a transformed Eulerian-mean approach to the eddy-mean-flow problem in z coordinates. It has become conventional to define the residual mean flow by adding to the streamfunction of the mean meridional flow a term like $\overline{v'b'}/\overline{b_z}$, where $\overline{v'b'}$ is the northward eddy

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flux of buoyancy and \bar{b}_z is its mean vertical gradient. This is the natural definition in quasigeostrophic theory, but it is in general just one of a range of possible choices. We find it preferable to use the coordinate-independent form introduced by Andrews and McIntyre (1978), in which the buoyancy flux and gradient are referenced to the local mean isopycnal slope. As will be described below, this form presents several advantages, both in handling the surface boundary condition and in the way eddy fluxes of buoyancy, potential vorticity, and momentum appear in the transformed equations for the mean state.

In what follows, we outline the underlying theory for the zonal-mean case. We pay particular attention to the “residual eddy fluxes”—the eddy fluxes as they appear in the transformed equations—including the fluxes of Ertel potential vorticity (PV) and angular momentum that are required to derive a closed theory for the mean state. Given our definition of the residual circulation, under conditions of small Rossby number (but arbitrary isopycnal slope) the eddy forcing of mean angular momentum is proportional to the component of the residual eddy PV flux along the mean isopycnals plus, near the surface, a term involving the diapycnal buoyancy flux.

2. Eulerian-mean equations

In conventional terms, the zonal-mean problem can be summarized as

$$\begin{aligned}\bar{u}_t + \bar{\mathbf{u}} \cdot \nabla \bar{m} &= -\nabla \cdot \mathbf{F}\{m\} + \bar{X}, \\ f\bar{u}_z &= -\bar{b}_y, \\ \bar{v}_y + \bar{w}_z &= 0, \quad \text{and} \\ \bar{b}_t + \bar{\mathbf{u}} \cdot \nabla \bar{b} &= -\nabla \cdot \mathbf{F}\{b\} + \bar{Q}.\end{aligned}\quad (1)$$

Here \mathbf{u} is the velocity, with components (u, v, w) in the (x, y, z) directions, $b = -g(\rho - \rho_0)/\rho_0$ is the buoyancy, $m = u - \int f dy$ the linear angular momentum per unit mass, X and Q represent sources and sinks of angular momentum and buoyancy, and subscripts denote partial differentiation. The overbar denotes the zonal (x) average, the prime quantities are departures from the mean, and the eddy fluxes are written in the form $\mathbf{F}\{c\} = \overline{\mathbf{u}'c'}$. Note that we have assumed thermal shear balance in the y direction; this is based not necessarily on small Rossby number, but on the assumed weakness of the meridional flow. Note also that \bar{u} is the zonal component of the mean flow, while we use the vector notation $\bar{\mathbf{u}}$ and \mathbf{F} to denote the meridional (northward and vertical) components only, since only these components appear in (1); we shall retain this convention throughout. The set (1), together with suitable boundary conditions, forms a closed set for the mean state variables \bar{u} , \bar{v} , \bar{w} , and \bar{b} , given the mean nonconservative terms \bar{X} and \bar{Q} , and the eddy fluxes of buoyancy and angular momentum.

3. The residual circulation and residual fluxes

The introduction of transformed Eulerian-mean (TEM) theory (Andrews and McIntyre 1976, 1978) greatly simplified analysis of the dynamics of the zonal-mean meridional structure and circulation of the atmosphere and ocean (e.g., Andrews et al. 1987; Haynes et al. 1991; Marshall and Radko 2003). The TEM equations corresponding to (1) hinge on the transformation $\bar{\mathbf{u}} \rightarrow \bar{\mathbf{u}}^\dagger$, where the “residual circulation” is

$$\bar{\mathbf{u}}^\dagger = \bar{\mathbf{u}} + \nabla \times \mathbf{i}\psi, \quad (2)$$

where \mathbf{i} is the unit vector in the x direction, and ψ , the “quasi-Stokes streamfunction,” is yet to be determined. Making this transformation amounts to a repartitioning of the fluxes, since the “mean” flux of a quantity of concentration c , which is $\bar{\mathbf{u}}\bar{c}$ in the conventional formulation, becomes $\bar{\mathbf{u}}^\dagger\bar{c}$ in TEM; the eddy flux is correspondingly, if sometimes implicitly, redefined in such a way that the divergence of the total flux (though not necessarily the flux itself) is unchanged. The underlying motivation for the transformation is the desire to identify those components of the eddy fluxes that are skew (i.e., directed normal to the mean gradient), and thus advective in nature, and to incorporate these components into the “mean” flux. In order thus to simplify the mean budgets (1), this procedure is applied to the buoyancy equation but, once the transformation (2) is chosen, other budgets must be transformed consistently to produce a closed set of equations for the mean state. One particularly important result of this procedure is that the momentum budget becomes more readily understood in terms of basic eddy properties, in the sense that the eddy forcing term in the transformed budget is directly dependent on downgradient fluxes of PV and buoyancy and thus represents nonconservative processes.

Consider a quantity c that satisfies a mean conservation equation of the form

$$c_t + \bar{\mathbf{u}} \cdot \nabla c = -\nabla \cdot \mathbf{F}\{c\} + \overline{S\{c\}}, \quad (3)$$

where $\mathbf{F}\{c\} = \overline{\mathbf{u}'c'}$ is the eddy flux of c , and $\overline{S\{c\}}$ represents mean sources and sinks. The quantity c could be buoyancy, PV, a chemical tracer, or, in a zonal-mean problem, absolute angular momentum density. Under the transformation (2), (3) becomes

$$c_t + \bar{\mathbf{u}}^\dagger \cdot \nabla c = -\nabla \cdot \mathbf{F}^\dagger\{c\} + \overline{S\{c\}}, \quad (4)$$

where we have introduced the “residual eddy flux”

$$\mathbf{F}^\dagger\{c\} = \mathbf{F}\{c\} - \psi \mathbf{i} \times \nabla c \quad (5)$$

(a nondivergent component has been removed). An expression of the form (5) was presented for trace chemicals by Andrews et al. (1987). Since the additional term involving ψ is skew (being directed along the \bar{c} contours), ψ can be chosen in such a way as to eliminate entirely the skew component of the original eddy flux,

leaving just the flux component along $\nabla\bar{c}$ to remain in $\bar{\mathbf{F}}^\dagger$, by defining

$$\psi = |\nabla\bar{c}|^{-2} \mathbf{i} \times \nabla\bar{c} \cdot \mathbf{F}\{c\}. \quad (6)$$

However, such a choice depends on c : if we choose ψ to eliminate the skew component of the buoyancy flux, the same transformation (with of course the same ψ) is not guaranteed to eliminate the skew flux of PV or of other tracers. However, since under near-conservative conditions the skew fluxes are manifestations of advection by the Stokes drift (e.g., Plumb 1979) it is to be expected that (5) will approximately eliminate the skew component of the flux for all near-conserved tracers (including PV, away from boundaries).

4. Eddy buoyancy fluxes and choice of the residual circulation

As illustrated in Fig. 1, we introduce the unit vectors up and across the mean buoyancy gradient, $\mathbf{n} = \nabla\bar{b}/|\nabla\bar{b}|$ and $\mathbf{s} = \mathbf{n} \times \mathbf{i}$ (so that \mathbf{s} is northward, along the mean isopycnals, for stable stratification), and separate the eddy buoyancy flux into components across and along the mean buoyancy gradient,

$$\mathbf{F}\{b\} = (\mathbf{s} \cdot \overline{\mathbf{u}'b'})\mathbf{s} + (\mathbf{n} \cdot \overline{\mathbf{u}'b'})\mathbf{n}.$$

Now, in order to remove entirely the skew eddy flux of buoyancy, we would simply implement (6) with $c \rightarrow b$, giving $\psi = -|\nabla\bar{b}|^{-1} \mathbf{s} \cdot \overline{\mathbf{u}'b'}$. However, we note that the transformed buoyancy budget retains the same general form under a more flexible definition of the residual circulation,

$$\psi \equiv -|\nabla\bar{b}|^{-1} [\mathbf{s} \cdot \overline{\mathbf{u}'b'} - \alpha(\mathbf{n} \cdot \overline{\mathbf{u}'b'})], \quad (7)$$

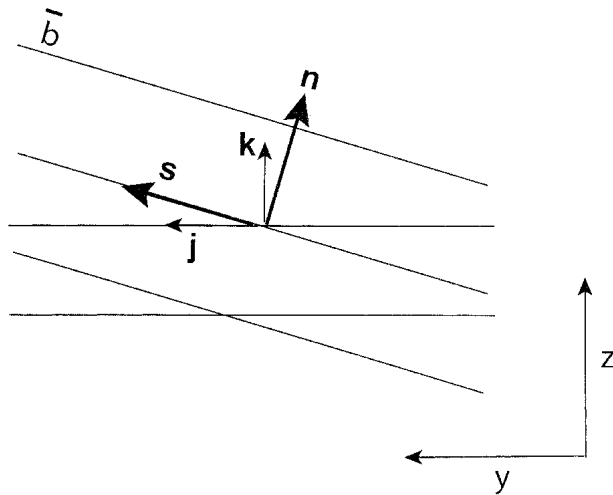


FIG. 1. Illustration of the unit vectors used in the analysis. The sloping lines are surfaces of constant mean buoyancy, ($\mathbf{i}, \mathbf{j}, \mathbf{k}$) are the unit vectors in the (x, y, z) directions (the vector \mathbf{i} is into the page); \mathbf{n} and $\mathbf{s} = \mathbf{n} \times \mathbf{i}$ are up the mean buoyancy gradient and along the mean buoyancy contours, respectively.

where α , which may be a function of space, is presently arbitrary. With this substitution in the last of (1) the mean buoyancy budget takes the form of (4), specifically

$$\bar{b}_t + (\bar{\mathbf{u}}^\dagger \cdot \nabla)\bar{b} = -\nabla \cdot \mathbf{F}^\dagger\{b\} + \bar{Q}, \quad (8)$$

when the residual buoyancy flux becomes, from (5),

$$\mathbf{F}^\dagger\{b\} = \overline{\mathbf{u}'b'} - \psi \mathbf{i} \times \nabla\bar{b} = (\mathbf{n} + \alpha\mathbf{s})\mathbf{n} \cdot \overline{\mathbf{u}'b'}. \quad (9)$$

The residual flux thus depends only on the diapycnal¹ component of the “raw” flux. For statistically steady eddies ($\partial\bar{b}^2/\partial t = 0$) under adiabatic conditions, the diapycnal eddy flux vanishes provided the eddies and meridional circulation are weak enough for advection of eddy buoyancy variance to be negligible (e.g., Treguier et al. 1997). Such is usually the case under zonal averaging (as we shall see confirmed in the example discussed in Part II); under time averaging, advection of eddy buoyancy variance can be approximately handled by a modification of the quasi-Stokes streamfunction (McDougall and McIntosh 1996). For the present case, it can be anticipated that $\mathbf{F}^\dagger\{b\}$ is negligibly small in the near-adiabatic conditions away from boundaries.

The diapycnal component of the residual flux in (9) is thus the same as that of the “raw” flux, but the isopycnal component depends on the choice of α . We will consider three special cases.

- (i) The first case is $\alpha = 0$. This choice, which we adopt here as our standard, was originally proposed for the nonquasigeostrophic case by Andrews and McIntyre (1978) and provides the cleanest separation in general of the diapycnal and isopycnal flux components. The transformation to the residual circulation corresponds to the definition (6) and involves the isopycnal flux alone:

$$\psi = -|\nabla\bar{b}|^{-1} (\mathbf{s} \cdot \overline{\mathbf{u}'b'}). \quad (10)$$

The residual eddy buoyancy flux,

$$\mathbf{F}^\dagger\{b\} = \mathbf{n}(\mathbf{n} \cdot \overline{\mathbf{u}'b'}), \quad (11)$$

is just the diapycnal component of the raw eddy flux, directed along the mean buoyancy gradient.

- (ii) The second case is $\alpha = -\bar{b}_y/\bar{b}_z$. With this choice of α (which equals the isopycnal slope), we have

$$\psi = -\frac{\overline{v'b'}}{\bar{b}_z}. \quad (12)$$

This is the form used by Andrews and McIntyre (1976) for the quasigeostrophic case and, in fact, cases i and ii are formally identical in the quasigeostrophic limit when isopycnal slopes are shallow. The corresponding residual eddy flux is

¹ By “diapycnal” and “isopycnal” fluxes, we mean with respect to the *mean* isopycnals throughout.

$$\mathbf{F}^\dagger\{b\} = \frac{|\nabla\bar{b}|}{\bar{b}_z} \mathbf{k}(\mathbf{n} \cdot \overline{\mathbf{u}'b'}), \quad (13)$$

where \mathbf{k} is the unit vector in the vertical direction.

- (iii) The third case is $\alpha = \bar{b}_z/\bar{b}_y$. This choice leads to the Held and Schneider (1999) definition of the residual circulation:

$$\psi = \frac{\overline{w'b'}}{\bar{b}_y}, \quad (14)$$

with the corresponding residual buoyancy flux

$$\mathbf{F}^\dagger\{b\} = \frac{|\nabla\bar{b}|}{\bar{b}_y} \mathbf{j}(\mathbf{n} \cdot \overline{\mathbf{u}'b'}), \quad (15)$$

where \mathbf{j} is the northward unit vector (at constant height).

So if, for example, the diapycnal eddy flux is down-gradient ($\mathbf{n} \cdot \overline{\mathbf{u}'b'} < 0$), the residual buoyancy flux is directed downgradient for case i, downward (assuming a statically stable mean state) for case ii, and horizontally, down the horizontal mean gradient, for case iii. The contribution of $\nabla \cdot \mathbf{F}^\dagger\{b\}$ to the mean budget (8) is formally negligible in cases i and ii under standard quasigeostrophic assumptions, but not in case iii. (In fact, if the isopycnal slope γ is small, the residual flux in case iii is of order γ^{-1} larger than in cases i and ii.) Of course, the residual flux is zero in all three cases wherever the diapycnal component of the raw eddy flux vanishes.

5. The surface boundary condition

The major differences among these cases show up near the surface, where diabatic effects are important and where the isopycnal slope may not be small. In fact, it is well known that the TEM transformation introduces difficulties with the surface boundary condition. In case ii, if there is any surface buoyancy gradient, there will be a nonzero raw diapycnal eddy flux at the surface and therefore the residual eddy flux, directed vertically, will usually be nonzero at the surface itself. Correspondingly (since the total buoyancy flux through the surface vanishes), there must be a compensating mean (residual) flux and therefore a nonzero residual mean flow through the surface. Such a flow is peculiar, from a conceptual point of view, and can be problematic in practice. One way around the problem is to exploit the fact that the residual flux through the surface disappears if buoyancy is uniform along the surface. One can then (e.g., Killworth 1997; Treguier et al. 1997; Held and Schneider 1999) postulate the existence of a thin boundary layer lying between the “real” fluid and this introduced surface; within this layer, the mean buoyancy contours are such as to make the transition between the buoyancy distribution at the real surface and uniform buoyancy at the hypothetical surface, as depicted in Fig. 2. This layer can be made to be (and

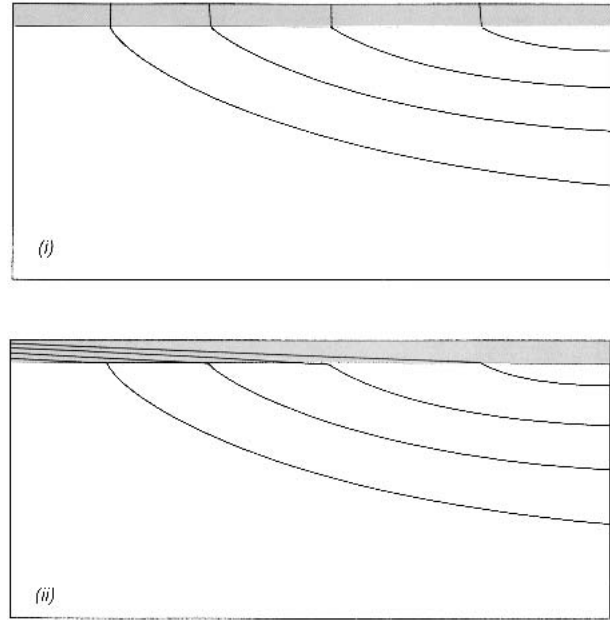


FIG. 2. Mean buoyancy contours within the hypothetical surface boundary layer (shaded) in cases (top) i and (bottom) ii. See text for discussion.

ideally should be) vanishingly thin; in the limit of vanishing thickness, it reduces in the quasigeostrophic case to the surface PV sheet introduced, for related reasons, by Bretherton (1966). The residual streamfunction makes a transition from its nonzero values at the top of the real fluid to a constant value (usually zero) at the hypothetical surface, and there is therefore a finite residual mass flux within the layer that closes off the circulation. Despite the rather artificial way the boundary layer has been introduced, this mass flux is nonetheless real, representing the finite isopycnal mass flux within the layer within which the isopycnals outcrop (Held and Schneider 1999). While the infinite static stability within the infinitesimally thin layer is no problem conceptually, it can pose difficulties in practice if such a layer is introduced in a numerical model, since the actual boundary conditions can become insulated from the fluid interior.

Held and Schneider (1999) proposed adopting the transformation of case iii to address this issue at a horizontal surface, since then the residual flux is necessarily horizontal and, by definition (14), the residual flow through the surface vanishes.² Because (14) may be difficult to evaluate in the interior where \bar{b}_y and $\overline{w'b'}$ may both be small (and because cases ii and iii become identical under the adiabatic conditions expected in the interior), they advocated defining ψ using definition

² Held and Schneider also noted that, even by definition i, the flux vanishes at a no-slip boundary where all components of the velocity vanish; adjustment to this boundary condition then takes place within the thin viscous boundary layer.

(12) in the interior and (14) near the surface, making some smooth transition from one to the other. The surface mass flux, confined to the thin boundary layer in case ii, is then spread out over the finite transition region.

To what extent case i is problematic depends on the structure of \bar{b} near the surface. In the presence of a mixed layer where $\bar{b}_z = 0$, \mathbf{s} becomes vertical and (10) becomes identical to (14) there.³ In fact, in this case definition (10) automatically (and objectively) makes the transition between (14) near the surface and (12) in the interior. If the absence of a mixed layer, however, the problem of residual flow through the surface remains. However, we can still postulate an impermeable surface slightly above the actual surface, as for case ii, and in fact doing so presents fewer difficulties. In case i, the condition for no normal flow at the surface is that the mean isopycnals are vertical there, and so the hypothetical boundary layer has zero static stability, as shown in Fig. 2. Therefore the modeling difficulties associated with infinite static stability of the layer in case ii do not apply here, since it can be represented by a thin, vertically well mixed layer that is transparent to information imposed at the upper boundary.

6. The TEM angular momentum budget and residual flux

In order to form a closed set of transformed mean budget equations analogous to (1), we need to transform all of the equations to replace the mean flow by the residual flow. The thermal wind equation is unaffected, and the same form of the continuity equation for the residual mean flow is guaranteed by the definition (2). Only the angular momentum equation is altered. Making the substitution $\bar{\mathbf{u}} \rightarrow \bar{\mathbf{u}}^\dagger = \bar{\mathbf{u}} + \nabla \times \mathbf{i}\psi$ in the first of (1), the transformed zonal-mean angular momentum budget becomes

$$\bar{u}_t + (\bar{\mathbf{u}}^\dagger \cdot \nabla) \bar{m} = -\nabla \cdot \mathbf{F}^\dagger\{m\} + \bar{X}, \quad (16)$$

where the residual flux of angular momentum, $\mathbf{F}^\dagger\{m\}$ is

$$\mathbf{F}^\dagger\{m\} = \mathbf{F}\{m\} - \psi \mathbf{i} \times \nabla \bar{m} = \overline{\mathbf{u}'v'} - \psi \bar{\zeta}_a, \quad (17)$$

where $\bar{\zeta}_a = \nabla \times (\mathbf{i}\bar{m})$ is the absolute vorticity of the mean zonal flow. In the standard TEM case ii, $\mathbf{F}^\dagger\{m\} = -\mathbf{E}$, the negative of the Eliassen–Palm flux, according to the standard definition (12) of the quasi-Stokes streamfunction.⁴

³ This assumes, however, that the isopycnals are truly vertical within the mixed layer. They will not normally be so in a global system with a small aspect ratio; however, this can be addressed by first stretching coordinates, to make the aspect ratio of order unity, before applying (10).

⁴ In fact, we could adopt the negative of (17) as a definition of Eliassen–Palm flux within any of definitions i, ii, or iii. Since the difference in $\mathbf{F}^\dagger\{m\}$ under definitions (11), (13), and (15) is a term that vanishes for small-amplitude eddies under adiabatic conditions, the form of the Eliassen–Palm relation (Andrews and McIntyre 1976) is unchanged.

7. The residual eddy PV flux

While separate consideration of the PV budget is not necessary to develop a complete set of transformed mean equations, it is of some interest to do so, and we shall have reason to invoke the residual eddy PV flux in what follows. With the residual circulation defined according to (10), the residual PV flux becomes, from (5) with $c \rightarrow P$,

$$\mathbf{F}^\dagger\{P\} = \overline{\mathbf{u}'P'} - \frac{\mathbf{s} \cdot \overline{\mathbf{u}'b'}}{|\nabla \bar{b}|} \mathbf{i} \times \nabla \bar{P}. \quad (18)$$

In general, the form (18) does not guarantee elimination of the skew component (as it would if ψ were based on P , rather than b). As is evident in the example discussed in Part II, however, in practice the skew component is largely eliminated in the adiabatic interior, as is to be expected theoretically on the basis of the following arguments.

The significance of the residual flux of tracers such as PV can be illustrated in some limiting cases. For small-amplitude eddies, the raw flux of a conservative tracer of concentration c can be written (e.g., Plumb 1979)

$$\mathbf{F}\{c\} = -\mathbf{K}\nabla \bar{c}, \quad (19)$$

where \mathbf{K} is a four-component diffusivity tensor whose components are functions of the eddy displacements and are the same for different conservative tracers. The components of $\mathbf{F}\{c\}$ and $\mathbf{F}^\dagger\{c\}$ along the mean isopycnals are

$$\mathbf{s} \cdot \mathbf{F}\{c\} = -K^{ss}\bar{c}_s - K^{sn}\bar{c}_n \quad \text{and}$$

$$\mathbf{s} \cdot \mathbf{F}^\dagger\{c\} = -K^{ss}\bar{c}_s - (K^{sn} - \psi)\bar{c}_n,$$

where \bar{c}_s and \bar{c}_n are the components of $\nabla \bar{c}$ in the s and n directions. If c is buoyancy, $\bar{c}_s \equiv \bar{b}_s = 0$ by construction, and so the choice (10) of $\psi = -\mathbf{s} \cdot \mathbf{F}\{b\}/\bar{b}_n = K^{sn}$ leaves $\mathbf{s} \cdot \mathbf{F}^\dagger\{b\} = 0$: the residual buoyancy flux is directed exclusively across the mean isopycnals. [In fact, as we have seen, this is the case whether or not (19) is appropriate.] More important for our purposes here, for other conserved quantities satisfying (19), such as PV or chemical tracers (but not angular momentum, which is not conserved within the eddies), we then have

$$\mathbf{s} \cdot \mathbf{F}^\dagger\{c\} = -K^{ss}\bar{c}_s; \quad (20)$$

that is, the component of the residual flux along the mean isentropes is nonzero, but is free of the skew-component, being independent of \bar{c}_n . Among other things, it follows that if one were to use a parameterization of this type, it is the residual eddy PV flux, and not the raw flux, that should be parameterized as a downgradient flux along the mean isopycnals. It also follows from (20) that it is the residual eddy PV flux, and not the raw flux, that vanishes if mean PV is homogenized along the mean isopycnals.

As a second illustration, in the quasigeostrophic limit, $P' \approx fb'_z + \zeta' \bar{b}_z$, and so

$$\overline{v'P'} \approx f \overline{v'b'_z} + \overline{v'\zeta'} \bar{b}_z.$$

The horizontal flux of quasigeostrophic PV is

$$\overline{v'q'} = \overline{v'\zeta'} + f v' \left(\frac{b'}{\bar{b}_z} \right)_z = \overline{v'\zeta'} + f \frac{\overline{v'b'_z}}{\bar{b}_z} - f \frac{\overline{v'b'}}{\bar{b}_z^2} \bar{b}_{zz},$$

whence

$$\overline{v'q'} = \bar{b}_z^{-1} \left[\overline{v'P'} - \left(\frac{\overline{v'b'}}{\bar{b}_z} \right) \bar{P}_z \right],$$

since, under the small isopycnal slope assumption of quasigeostrophic theory, $f \bar{b}_{zz} \approx \bar{P}_z$. For case ii, under definition (12) of the quasi-Stokes streamfunction, then, (18) gives

$$\overline{v'q'} \approx \bar{b}_z^{-1} F_y^\dagger \{P\}.$$

Thus, the horizontal eddy flux of quasigeostrophic PV is proportional to the *residual* flux of Ertel PV, rather than to the raw flux, in case ii. In fact, since definitions (10) and (12) are indistinguishable in the quasigeostrophic limit, the same is true for case i. The same is not true, however, for case iii under definition (14), except of course in adiabatic situations where cases i–iii are indistinguishable.

8. Relationship between residual fluxes of buoyancy, PV, and angular momentum

The complete transformed mean problem is defined by (8) and (16), together with the balance and continuity equations:

$$\begin{aligned} \bar{u}_t + \bar{\mathbf{u}}^\dagger \cdot \nabla \bar{m} &= -\nabla \cdot \mathbf{F}^\dagger \{m\} + \bar{X}, \\ f \bar{u}_z &= -\bar{b}_y, \\ \bar{v}_y^\dagger + \bar{w}_z^\dagger &= 0, \quad \text{and} \\ \bar{b}_t + \bar{\mathbf{u}}^\dagger \cdot \nabla \bar{b} &= -\nabla \cdot \mathbf{F}^\dagger \{b\} + \bar{Q}. \end{aligned} \quad (21)$$

Since in our case the balance equation is devoid of eddy terms, the eddy forcing comprises two terms: the momentum forcing $-\nabla \cdot \mathbf{F}^\dagger \{m\}$ and the buoyancy forcing $-\nabla \cdot \mathbf{F}^\dagger \{b\}$. In structural terms, nothing has changed from the basic set (1): we have simply replaced \mathbf{F} by \mathbf{F}^\dagger for the two fluxes and redefined the mean circulation. However, whether one is interested in parameterizing the eddy forcing in a numerical model, or in theoretical analysis of the problem, there are several advantages to the transformed set. First, in the transformed set (21) only the downgradient component of the eddy buoyancy flux remains in the buoyancy forcing, and its form will be at least qualitatively clear in many problems of interest. The more difficult skew component, which unlike the downgradient component is not directly related to eddy dissipation and stirring, does not need to be explicitly specified, since it has been absorbed into the

definition of the residual circulation and is a part of the response (i.e., it has become part of the solution) rather than of the forcing.

A very important second property of the corresponding quasigeostrophic set is that eddy forcing of the angular momentum equation can be expressed as the northward quasigeostrophic PV flux, or equivalently as the divergence of the Eliassen–Palm flux, which provides, through linkage with eddy sources, sinks, and propagation, some qualitative constraints on its form that can be deduced from a knowledge of the dynamics of the eddies. It is of some interest, therefore, to establish whether the “eddy drag”—the divergence of the residual eddy flux of momentum—can be expressed in terms of the residual eddy fluxes of buoyancy and PV in the more general case.

In general, such a relationship is complex. It is shown in appendix A that

$$\begin{aligned} \mathbf{F}^\dagger \{P\} &= (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot \mathbf{F}^\dagger \{m\} + (\bar{\zeta}_a \cdot \nabla) \mathbf{F}^\dagger \{b\} \\ &\quad - (\mathbf{F}^\dagger \{b\} \cdot \nabla) \bar{\zeta}_a - (\bar{\zeta}' b')_t + (\mathbf{i} \times \nabla \psi) (\bar{\zeta}' \cdot \nabla) \bar{b}' \\ &\quad + \nabla \times (\bar{\mathbf{u}}' b' \times \bar{\zeta}' - \mathbf{i} \times \nabla \psi \times \bar{\zeta}' b') \\ &\quad - \mathbf{i} \times \bar{b}' \nabla \bar{X}' + \bar{Q}' \bar{\zeta}'. \end{aligned} \quad (22)$$

The curl term is of course nondivergent and plays no role in the mean PV budget, but it is not generally negligible for our purposes, since what is needed is a relationship involving the flux itself, rather than just its divergence.

Relationships for small Rossby number

It is shown in appendix B that, for small Rossby number R (but without any assumption of small isopycnal slope, nor of small amplitude eddies, beyond what is implied by $R \ll 1$), (22) reduces to

$$\mathbf{F}^\dagger \{P\} = (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot \mathbf{F}^\dagger \{m\} + f \frac{\partial}{\partial z} \mathbf{F}^\dagger \{b\} - \mathbf{k} \frac{df}{dy} F^{\dagger(y)} \{b\}. \quad (23)$$

This relationship allows specification of the divergence of the residual angular momentum flux—the required forcing term for the TEM momentum equation—in terms of fluxes of the conserved quantities b and P .

Now, under adiabatic conditions for which $\mathbf{F}^\dagger \{b\} = 0$,

$$\mathbf{F}^\dagger \{P\} = (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot \mathbf{F}^\dagger \{m\};$$

the residual eddy PV flux is then directed exclusively along the buoyancy contours (or is zero). Further, in the mean momentum equation the eddy forcing term becomes simply

$$\nabla \cdot \mathbf{F}^\dagger \{m\} = -|\nabla \bar{b}|^{-1} \mathbf{s} \cdot \mathbf{F}^\dagger \{P\}. \quad (24)$$

In the adiabatic, small-Rossby-number case, therefore, the eddy momentum forcing is simply proportional to the isopycnal component of the residual eddy flux of PV. Note that (24) is, to a point, independent of the

definition of ψ . It is therefore valid, not just for the three possible choices for ψ discussed in section 4, but also for definitions (e.g., McDougall and McIntosh 1996) modified to accommodate the advection of buoyancy variance. However, the result presumes that the scaling analysis of appendix B is valid, a condition that precludes $\psi = 0$, for example, so that (24) is not valid for the “raw” eddy fluxes.

In adiabatic regions, therefore, a term proportional to the isopycnal component of the residual eddy PV flux entirely describes the eddy forcing of the mean state. However, in other regions, especially near the surface within an important region we shall discuss at length in Part II and which we call the “surface diabatic layer,” the residual eddy flux of buoyancy becomes significant and appears in the mean state equations both directly, via the mean buoyancy equation, and indirectly, via the additional terms in (23).

Whether or not conditions are adiabatic, in the “standard” TEM case ii, $\psi = -\bar{b}_z^{-1}\bar{v}'\bar{b}'$ and $\mathbf{F}^\dagger\{b\}$, given by (13), is directed purely vertically and so we may take the horizontal component of (23) to give simply

$$\nabla \cdot \mathbf{F}^\dagger\{m\} = -\bar{b}_z^{-1}F^{\dagger(y)}\{P\}. \quad (25)$$

This is in fact the same form that is applicable in the quasigeostrophic case [see appendix B, (B2)].

For case iii, (15), $\mathbf{F}^\dagger\{b\}$ is directed purely in the y direction. In this case, it is not apparent that taking any single component of (23) yields a useful expression for $\nabla \cdot \mathbf{F}^\dagger\{m\}$.

For case i, $\alpha = 0$, $\psi = -|\nabla\bar{b}|^{-1}(\mathbf{s} \cdot \bar{\mathbf{u}}'b')$, and $\mathbf{F}^\dagger\{b\} = (\bar{\mathbf{u}}'b' \cdot \mathbf{n})\mathbf{n}$ is directed normal to the isopycnals. If we take the \mathbf{s} component of (23) and use the identity

$$\mathbf{n}_z = \frac{\bar{b}_z^2}{|\nabla\bar{b}|^2} \left(\frac{\bar{b}_y}{\bar{b}_z} \right)_z \mathbf{s},$$

then we have

$$\begin{aligned} \nabla \cdot \mathbf{F}^\dagger\{m\} = & -|\nabla\bar{b}|^{-1} \left\langle F^{\dagger(s)}\{P\} - \left[f \frac{\bar{b}_z^2}{|\nabla\bar{b}|^2} \left(\frac{\bar{b}_y}{\bar{b}_z} \right)_z \right. \right. \\ & \left. \left. - \frac{df}{dy} \frac{\bar{b}_y^2}{|\nabla\bar{b}|^2} \right] F^{\dagger(n)}\{b\} \right\rangle, \quad (26) \end{aligned}$$

relating the angular momentum flux divergence to the PV flux along the mean isopycnals and the diapycnal buoyancy flux. Now, the additional terms involving $F^{\dagger(n)}\{b\}$ are significant only within a relatively thin surface diabatic layer, of depth δ , say. It is shown in appendix B that the ratio of the second to the first of the two terms in the square bracket in (26) is $\hat{\beta}H^2/L^2$, where H and L are relevant height and length scales, and $\hat{\beta} = (L/f)df/dy \sim L/a$, where a is the Earth radius. Since $\hat{\beta} \leq O(1)$, the term involving β is negligible provided $H \approx \delta \ll L$. Further (see appendix B, again), the ratio of the term involving $F^{\dagger(n)}\{b\}$ to that involving $F^{\dagger(s)}\{P\}$ is of order ε^2 (where ε is the isopycnal slope);

this term is only of significance, therefore, when $\varepsilon \sim O(1)$. In that case, PV may be represented by its “planetary geostrophic” form $P \approx fb_z$, when

$$\mathbf{s} \cdot \nabla\bar{P} = |\nabla\bar{b}|^{-1}(\bar{b}_z\bar{P}_y - \bar{b}_y\bar{P}_z) = \frac{\bar{b}_z^2}{|\nabla\bar{b}|} \left[f \left(\frac{\bar{b}_y}{\bar{b}_z} \right)_z + \frac{df}{dy} \right].$$

Hence, (26) can be written

$$\begin{aligned} \nabla \cdot \mathbf{F}^\dagger\{m\} = & -|\nabla\bar{b}|^{-1} \\ & \times \left[F^{\dagger(s)}\{P\} - \left(|\nabla\bar{b}|^{-1}\mathbf{s} \cdot \nabla\bar{P} - \frac{df}{dy} \right) F^{\dagger(n)}\{b\} \right]. \end{aligned}$$

Again, the term involving df/dy is negligible for a thin layer, and

$$\begin{aligned} \nabla \cdot \mathbf{F}^\dagger\{m\} = & -|\nabla\bar{b}|^{-1} [F^{\dagger(s)}\{P\} \\ & - |\nabla\bar{b}|^{-1}(\mathbf{s} \cdot \nabla\bar{P}) F^{\dagger(n)}\{b\}] \quad (27) \end{aligned}$$

is a satisfactory approximation to (26) valid for $R \ll 1$ and for any ε . Note that the additional term is proportional to the mean isopycnal PV gradient $\mathbf{s} \cdot \nabla\bar{P}$ and so if the residual PV flux is parameterized using a flux-gradient relation, the entire right-hand side of (27) is so proportional, in which case the addition of the diabatic term has served only to modify the factor multiplying $\mathbf{s} \cdot \nabla\bar{P}$.

9. Conclusions

The coordinate-independent definition of the residual circulation adopted here is imperceptibly different from the conventional definition wherever the mean isopycnal slope is small, which will frequently mean everywhere except in the diabatic surface layer, where it affords some important advantages. One such advantage is that, in the presence of a mixed layer, the residual circulation naturally satisfies a condition of no normal motion at the upper boundary. Even in the absence of a mixed layer, it is straightforward to impose one—however thin—to satisfy this condition, a procedure that we argue is preferable to imposing a thin layer of infinite static stability that is necessary to achieve the same end in the conventional approach. In fact, this formulation objectively achieves a near-surface transition like that advocated by Treguier et al. (1997); ψ evolves objectively from the Held and Schneider (1999) definition $-w'b'/\bar{b}_y$ at the surface (where the upper-boundary condition thus becomes the simple one of zero normal flow in the residual mean) and the better behaved $\bar{v}'b'/\bar{b}_z$ in the adiabatic interior. Indeed, the present formulation leads us to follow Treguier et al. (1997) in summarizing the effect of eddies as PV transport along mean isopycnals in adiabatic interior regions, plus diapycnal buoyancy transport in the diabatic surface layer. (Just how important this near-surface diabatic transport can be will be illustrated in Part II.) This view survives into the nongeostrophic case, even if the

appropriate expression for momentum transport becomes straightforward only for small Rossby number (though with finite isopycnal slopes).

We have stressed the importance of the distinction in z coordinates between “residual” eddy fluxes—the eddy fluxes as they appear in the transformed mean equations—and the “raw” fluxes. For example, the PV flux $\overline{\mathbf{u}'P'}$ (in these z coordinates) does not vanish even when the mean PV gradient along the mean isopycnals vanishes, because of a presence of a the “skew” component (e.g., see Fig. 1b of Part II). This distinction does not arise when the PV transport is two-dimensional: the horizontal flux of quasigeostrophic potential vorticity, and the flux of PV in isopycnal coordinates under adiabatic conditions, both vanish under such circumstances. Among other things, this implies that in any parameterization scheme that is centered on the eddy flux of (Ertel) PV, it is the residual flux that appears to be most amenable to parameterization via a flux–gradient formalism. It also means that, in practice, one should not be surprised to find little relation between raw fluxes of Ertel PV and the mean isopycnal gradient, even in circumstances in which the advection of buoyancy variance (McDougall and McIntosh 1996) is unlikely to be a serious issue.

More generally, while we have not pursued the details here, the theoretical formulation developed in this paper has important implications for approaches to eddy parameterization. As noted above, parameterization of the residual eddy flux of PV everywhere, and of the (diabatic) residual eddy buoyancy flux within the surface layer is required, consistent with the arguments of Treguier et al. (1997) for the quasigeostrophic case and of Held and Schneider (1999) in isopycnal coordinates. For implementation in models in which momentum (or vorticity), rather than PV, is the predicted vari-

able the discussion of section 8 shows how the eddy fluxes of momentum should be incorporated. Under adiabatic conditions, the result that the eddy forcing of the momentum equation is proportional to the residual eddy PV flux along the mean isopycnals parallels the quasigeostrophic and isopycnal-coordinate results. Within the diabatic surface layer, there are the additional terms but, for small Rossby number, these can easily be assimilated into a flux–gradient parameterization for PV transfer.

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APPENDIX A

Relationships between Residual PV, Buoyancy, and Momentum Fluxes

The Ertel PV perturbation is $P' = P - \bar{P} = \bar{\xi}_a \cdot \nabla b' + \xi' \cdot \nabla b + \xi' \cdot \nabla b' - \bar{\xi}' \cdot \nabla \bar{b}'$, and so

$$\overline{\mathbf{u}'P'} = \overline{\mathbf{u}'(\xi' \cdot \nabla)\bar{b}} + \overline{\mathbf{u}'(\bar{\xi}_a \cdot \nabla)b'} + \overline{\mathbf{u}'(\xi' \cdot \nabla)b'}. \quad (\text{A1})$$

Now, the perturbations equations can be written

$$\begin{aligned} u'_t - \bar{\xi}_a v' + \bar{\eta} w' + \phi'_x &= \tilde{X}, \\ v'_t + \bar{\xi}_a u' + \bar{u} \xi' + \phi'_y &= \tilde{Y}, \\ w'_t - u' \bar{\eta} - \bar{u} \eta' + \phi'_z &= b' + \tilde{Z}, \quad \text{and} \\ b'_t + \bar{u} b'_x + v' \bar{b}_y + w' \bar{b}_z &= \tilde{Q}, \end{aligned} \quad (\text{A2})$$

where $\phi = \rho^{-1} p + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}$ is the Bernoulli function, and for convenience we have absorbed the nonlinear combinations of perturbation quantities into the nonconservative terms:

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \\ \tilde{Q} \end{pmatrix} = \begin{pmatrix} X' + v' \xi' - w' \eta' + \bar{v} \xi' - \bar{w} \eta' \\ Y' - u' \xi' + w' \xi' + \bar{w} \xi' - w' \bar{\xi} \\ Z' + u' \eta' - v' \xi' - \bar{v} \xi' - v' \bar{\xi} \\ Q' - u' b'_x - v' b'_y - w' b'_z - \bar{v} b'_y - \bar{w} b'_z \end{pmatrix}, \quad (\text{A3})$$

where (X, Y, Z, Q) are the relevant nonconservative sources and sinks. Then, after some manipulation involving (A1) and (A2),

$$\begin{aligned} \overline{v'P'} &= \bar{b}_z (\overline{v' \xi'} - \overline{w' \eta'}) + \bar{\eta} \nabla \cdot \overline{\mathbf{u}' b'} - \overline{\eta' b'_t} + \overline{u'_t b'_z} \\ &+ \overline{\phi'_x b'_z} - \overline{\eta u' b'_x} - \overline{u \eta' b'_x} + \overline{\eta' \tilde{Q}} - \overline{\tilde{X} b'_z} \\ &+ \overline{v' (\xi' \cdot \nabla) b'} + \overline{v' b'_x \xi'}. \end{aligned}$$

However, using (A2) and the fact that $\overline{(\quad)_x} = 0$,

$$\begin{aligned} \text{(i)} \quad -\overline{\eta' b'_t} + \overline{u'_t b'_z} &= \overline{w'_x b'_t} - \overline{u'_z b'_t} + \overline{u'_t b'_z} \\ &= \overline{w'_x b'_t} - \overline{(u' b'_t)_z} + \overline{u' b'_{zt}} + \overline{u'_t b'_z} \\ &= -\overline{w' b'_{xt}} - \overline{(u' b'_t)_z} + \overline{(u' b'_z)_t}, \\ \text{(ii)} \quad \overline{\phi'_x b'_z} &= \overline{(\phi'_x b')_z} - \overline{\phi'_{xz} b'} = \overline{(\phi'_x b')_z} + \overline{\phi'_z b'_x}, \quad \text{and} \\ \text{(iii)} \quad \overline{w'_t b'_x} - \overline{\eta u' b'_x} - \overline{u \eta' b'_x} + \overline{\phi'_z b'_x} &= \overline{b' b'_x} + \overline{\tilde{Z} b'_x} = \overline{\tilde{Z} b'_x}. \end{aligned}$$

and so

$$\begin{aligned} \overline{v'P'} &= \overline{b_z}(\overline{v'\zeta'} - \overline{w'\eta'}) + \overline{\eta} \nabla \cdot \overline{\mathbf{u}'b'} - (\overline{u'b'_t} - \overline{\phi'_x b'})_z \\ &\quad + (\overline{u'b'_z} - \overline{w'b'_x})_t + \overline{\eta' \tilde{Q}} - \overline{\tilde{X}b'_z} + \overline{\tilde{Z}b'_x}. \end{aligned} \quad (\text{A4})$$

Similarly, for the z component,

$$\begin{aligned} \overline{w'P'} &= -\overline{b_y}(\overline{v'\zeta'} - \overline{w'\eta'}) + \overline{\zeta_a} \nabla \cdot \overline{\mathbf{u}'b'} - \overline{\zeta' b'_t} \\ &\quad - \overline{b'_y u'_t} - \overline{b'_y \phi'_x} - \overline{\zeta_a u' b'_x} - \overline{u' \zeta' b'_x} + \overline{\zeta' \tilde{Q}} + \overline{\tilde{X} b'_y} \\ &\quad - \overline{\tilde{Y} b'_x} + \overline{\xi w' b'_x}, \end{aligned}$$

and using

$$\begin{aligned} \text{(iv)} \quad -\overline{\zeta' b'_t} - \overline{b'_y u'_t} &= -\overline{v'_x b'_t} + \overline{u'_y b'_t} - \overline{b'_y u'_t} \\ &= (\overline{u' b'_y})_y - (\overline{u' b'_y})_t + \overline{v' b'_{xt}}, \end{aligned}$$

$$\text{(v)} \quad \overline{b'_y \phi'_x} = (\overline{\phi'_x b'})_y - \overline{\phi'_{xy} b'} = (\overline{\phi'_x b'})_y + \overline{\phi'_y b'_x}, \quad \text{and}$$

$$\text{(vi)} \quad \overline{v'_x b'_t} + \overline{\phi'_y b'_x} + \overline{\zeta_a u' b'_x} + \overline{u' \zeta' b'_x} = \overline{\tilde{Y} b'_x},$$

we have

$$\begin{aligned} \overline{w'P'} &= -\overline{b_y}(\overline{v'\zeta'} - \overline{w'\eta'}) + \overline{\zeta_a} \nabla \cdot \overline{\mathbf{u}'b'} + (\overline{u'b'_t} - \overline{\phi'_x b'})_y \\ &\quad - (\overline{u'b'_y} - \overline{v'b'_x})_t + \overline{\zeta' \tilde{Q}} + \overline{\tilde{X} b'_y} - \overline{\tilde{Y} b'_x} \\ &\quad + \overline{w'(\zeta' \cdot \nabla) b'}. \end{aligned} \quad (\text{A5})$$

In total, then, (A4) and (A5) give

$$\begin{aligned} \overline{\mathbf{u}'P'} &= (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot (\overline{\mathbf{u}'u'}) + \overline{\zeta_a} \nabla \cdot \overline{\mathbf{u}'b'} \\ &\quad + (\mathbf{i} \times \nabla) (\overline{u'b'_t} - \overline{\phi'_x b'}) - (\overline{\mathbf{u}' \times \nabla b'})_t \\ &\quad + \overline{\zeta' \tilde{Q}} + \overline{\tilde{\mathbf{X}} \times \nabla b'} + \overline{\mathbf{u}'(\zeta' \cdot \nabla) b'}, \end{aligned} \quad (\text{A6})$$

where $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{Y}, \tilde{Z})$. The third term on the right-hand side of (A6) is nondivergent, and the fourth is zero for statistically steady eddies.

Consider now the sum of the third and fourth terms of (A6), $\mathbf{T}_1 = (\mathbf{i} \times \nabla) (\overline{u'b'_t} - \overline{\phi'_x b'}) + (\overline{\mathbf{u}' \times \nabla b'})_t$. This has components

$$\begin{aligned} T_1^{(y)} &= -(\overline{u'b'_t} - \overline{\phi'_x b'})_z + (\overline{u'b'_z} - \overline{w'b'_x})_t \\ &= (\overline{v'b'_t} \zeta_a - \overline{w'b'_t \eta})_z - (\overline{u'_z b'} - \overline{w'_x b'})_t + (\overline{\tilde{X} b'})_z \end{aligned}$$

and

$$\begin{aligned} T_1^{(z)} &= (\overline{u'b'_t} - \overline{\phi'_x b'})_y - (\overline{u'b'_y} - \overline{v'b'_x})_t \\ &= -(\overline{v'b'_t} \zeta_a - \overline{w'b'_t \eta})_y + (\overline{u'_y b'} - \overline{v'_x b'})_t - (\overline{\tilde{X} b'})_y, \end{aligned}$$

where the first of (A2) has been used. Therefore $\mathbf{T}_1 = \nabla \times (\overline{\mathbf{u}'b'} \times \overline{\zeta_a} + \mathbf{i} \tilde{X} b') - (\overline{\zeta' b'})_t$, and so

$$\begin{aligned} \overline{\mathbf{u}'P'} &= (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot (\overline{\mathbf{u}'u'}) + \overline{\zeta_a} \nabla \cdot \overline{\mathbf{u}'b'} \\ &\quad - \nabla \times (\overline{\mathbf{u}'b'} \times \overline{\zeta_a}) - (\overline{\zeta' b'})_t + \overline{\zeta' \tilde{Q}} + \overline{b' \nabla \times \tilde{\mathbf{X}}} \\ &\quad + \overline{\mathbf{u}'(\zeta' \cdot \nabla) b'} + \overline{\xi \mathbf{u}' b'_x}. \end{aligned} \quad (\text{A7})$$

Let us now remove the nonlinear terms from the non-conservative terms in (A7), using (A3). The triple correlations give a total of

$$\mathbf{T}_2 = -\nabla \times \mathbf{i} [(\overline{v'\zeta'} - \overline{w'\eta'}) b']. \quad (\text{A8})$$

The contribution from the mean advection terms in (A3) is

$$\mathbf{T}_3 = \nabla \times \mathbf{i} (\overline{w\eta' b'} - \overline{v\zeta' b'}) - \overline{\mathbf{u}\zeta' \cdot \nabla b'} + \overline{\xi \mathbf{v}'_x b'}. \quad (\text{A9})$$

The last term just cancels the term $\overline{\xi \mathbf{v}'_x b'}$ in (A7), leaving, in total,

$$\begin{aligned} \mathbf{F}\{P\} &= \overline{\mathbf{u}'P'} = (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot (\overline{\mathbf{u}'u'}) + \overline{\zeta_a} \nabla \cdot \overline{\mathbf{u}'b'} \\ &\quad - \nabla \times (\overline{\mathbf{u}'b'} \times \overline{\zeta_a}) - (\overline{\zeta' b'})_t - \overline{\mathbf{u}\zeta' \cdot \nabla b'} \\ &\quad - \mathbf{i} \times \nabla [(\overline{v'\zeta'} - \overline{w'\eta'}) b'] \\ &\quad + \mathbf{i} \times \nabla (\overline{w\eta' b'} - \overline{v\zeta' b'}) + \overline{\zeta' \tilde{Q}} + \overline{b' \nabla \times \tilde{\mathbf{X}}}. \end{aligned} \quad (\text{A10})$$

Using the definitions (9), (18), and (17) for the residual fluxes, we finally arrive at (22).

APPENDIX B

Relationships for Small Rossby Number

If we neglect the nonconservative terms in (22) (thus assuming them to be weak at leading order), we have

$$\begin{aligned} \mathbf{F}^\dagger\{P\} &= (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot \mathbf{F}^\dagger\{m\} + (\overline{\zeta_a} \cdot \nabla) \mathbf{F}^\dagger\{b\} \\ &\quad - (\mathbf{F}^\dagger\{b\} \cdot \nabla) \overline{\zeta_a} - (\overline{\zeta' b'})_t + (\mathbf{i} \times \nabla \psi) \overline{\zeta' \cdot \nabla b'} \\ &\quad + \nabla \times (\overline{\mathbf{u}'b'} \times \overline{\zeta'} - \mathbf{i} \times \nabla \psi \times \overline{\zeta' b'}). \end{aligned} \quad (\text{B1})$$

We shall assume that the Rossby number R is small but that the isopycnal slope is not necessarily so. In fact, if we adopt the scalings H and L for vertical and horizontal length scales, respectively (assuming these apply to both eddies and mean state), and Γ for \bar{b}_z then $b_y \sim \varepsilon H/L\Gamma$ where ε is unrestricted (the case $\varepsilon \ll R$ being the quasigeostrophic limit). Further, even though we have not yet defined ψ , we assume (with the definitions of section 4 in mind) that ψ scales as $|\overline{\mathbf{u}'b'}|/|\nabla \bar{b}|$. Then the scalings for the horizontal components of the six terms on the right-hand side of (B1) are

$$f^2 L \Gamma \left(R\varepsilon, \frac{H^2}{L^2} R\varepsilon^3, \frac{H^2}{L^2} R\varepsilon^3 \hat{\beta}, R^2 \varepsilon^2, R^2 \varepsilon^2, R^2 \varepsilon \right),$$

where $\hat{\beta} = L\beta/f = L/f \, d\bar{b}/dy$. The corresponding scalings for the vertical components are

$$f^2 H \Gamma \left(R\varepsilon^2, R\varepsilon^2, \frac{H^2}{L^2} R\varepsilon^3 \hat{\beta}, R^2 \varepsilon, R^2 \varepsilon^2, R^2 \varepsilon \right).$$

In the quasigeostrophic case, $\varepsilon \ll R$, the single dominant term is the y component of the first, the other terms being at most $O(R)$ smaller, leaving

$$F^{\dagger(y)}\{P\} = -\bar{b}_z \nabla \cdot \mathbf{F}^{\dagger}\{m\}. \quad (\text{B2})$$

More generally, if we allow ε to be as large as $O(1)$, we must retain the first three terms, though we need only retain the leading-order contributions; specifically, replacing $\bar{\zeta}_a$ by $f\mathbf{k}$ leaves

$$\mathbf{F}^{\dagger}\{P\} = (\mathbf{i} \times \nabla \bar{b}) \nabla \cdot \mathbf{F}^{\dagger}\{m\} + f \frac{\partial}{\partial z} \mathbf{F}^{\dagger}\{b\} - \mathbf{k} \frac{df}{dy} F_y^{\dagger}\{b\}. \quad (\text{B3})$$

Note that the last term is significant only if $\hat{\beta} \sim O(1)$, that is, if the length scale is planetary.

Note also, in the context of the discussion in section 8a, following (26), that if (B3) is projected onto the s direction, the ratios of the second and third terms on the right-hand side to the first are ε^2 and $\varepsilon^2 \hat{\beta} H^2 / L^2$, respectively, assuming $H/L \leq O(1)$ and $\varepsilon \leq O(1)$.

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