# Radiation and Dissipation of Internal Waves Generated by Geostrophic Motions Impinging on Small-Scale Topography: Theory

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#### ABSTRACT

Observations and inverse models suggest that small-scale turbulent mixing is enhanced in the Southern Ocean in regions above rough topography. The enhancement extends O(1) km above the topography, suggesting that mixing is supported by the breaking of gravity waves radiated from the ocean bottom. In this study, it is shown that the observed mixing rates can be sustained by internal waves generated by geostrophic motions flowing over bottom topography. Weakly nonlinear theory is used to describe the internal wave generation and the feedback of the waves on the zonally averaged flow. Vigorous inertial oscillations are driven at the ocean bottom by waves generated at steep topography. The wave radiation and dissipation at equilibrium is therefore the result of both geostrophic flow and inertial oscillations differing substantially from the classical lee-wave problem. The theoretical predictions are tested versus two-dimensional high-resolution numerical simulations with parameters representative of Drake Passage. This work suggests that mixing in Drake Passage can be supported by geostrophic motions impinging on rough topography rather than by barotropic tidal motions, as is commonly assumed.

# 1. Introduction

Turbulent mixing plays an important role in the circulation of the Southern Ocean (SO). Observations of velocity and density fluctuations show that mixing is strongly enhanced above rough bottom topography (Naveira Garabato et al. 2004). Inverse calculations (Ganachaud and Wunsch 2000; Sloyan and Rintoul 2001) find that this vigorous turbulent mixing contributes crucially to the downward buoyancy flux that maintains the abyssal ocean stratification and to the upward transport of the waters that close the ocean's meridional overturning circulation. It is an open question as to what physics drives the enhanced mixing.

Polzin et al. (1995) show that turbulent mixing in the ocean interior away from the surface and bottom boundary layers is typically associated with breaking internal waves. In particular, they show that the intensity of turbulent fluctuations is well correlated with the local in-

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ternal wave activity. Gregg (1989) uses the correlation to parameterize the levels of turbulent mixing in terms of the background oceanic internal wave spectrum described by the Garrett-Munk (GM) empirical formula (Munk 1981). Parameterizations based on the GM internal wave spectral level have remarkable skill in predicting the background turbulent mixing found in most of the ocean, but they fail to characterize regions of enhanced mixing. The diapycnal mixing inferred by Naveira Garabato et al. (2004) above rough topography in the SO exceeds background values by one-three orders of magnitude. The vertically integrated dissipation rate averaged for a section across Drake Passage is of the order of 10 mW  $m^{-2}$  corresponding to a bottom diapycnal diffusivity of  $10^{-2}$  m<sup>2</sup> s<sup>-1</sup>, as opposed to background values of  $10^{-5}$  m<sup>2</sup> s<sup>-1</sup> found in the ocean thermocline. Note that the relationship between internal wave activity and mixing holds. The enhancement is associated with an increased internal wave activity over the GM background value. Kunze et al. (2006), using a similar finescale parameterization, also find enhanced mixing in regions with rough topography and strong bottom flows; however, their values of energy

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dissipation rate in the Southern Ocean are somewhat lower.

The enhancement of turbulent mixing above rough topography has been linked to the generation of internal waves by flows impinging on topography (Polzin et al. 1997). The generation of this wave activity would add to the background wave field and explain the enhancement of internal wave energy and associated mixing, but the question of what motions drive the bulk of the internal wave radiation remains unanswered. Nowlin et al. (1986) show, based on moored observations in Drake Passage, that the kinetic energy in the abyss is partitioned among geostrophic flows, inertial oscillations (IOs), tides, and the internal wave continuum. Any one of these motions can generate internal waves through interaction with bottom topography; however, most of the recent theoretical work has focused on barotropic tides (e.g., Garrett and St. Laurent 2002; Garrett and Kunze 2007). Internal wave generation by the dominant  $M_2$  tide component is estimated globally by Nycander (2005) using the linear theory of internal wave generation developed by Bell (1975a,b) and modified by Llewellyn Smith and Young (2002) to account for the finite ocean depth. Using this estimate, we find that the energy flux radiated by internal waves from topography deeper than 2 km in the Drake Passage region is at most 1-2 mW m<sup>-2</sup>. In the region of the mid-Atlantic Ridge tidal flows have been shown to radiate mostly at low vertical modes and only a small fraction, less than 30%, goes into high modes that can dissipate locally (St. Laurent and Garrett 2002). Low modes, accounting for the bulk of energy radiation, can either be influenced by topography and scatter their energy to higher wavenumbers that dissipate locally or radiate away and contribute to mixing in remote locations (St. Laurent and Garrett 2002). Even if all radiated energy dissipated locally, tidal flows in Drake Passage can account for no more than 10%-20% of the observed dissipation rates.

Bottom geostrophic flows are much more intense in the SO than in most other ocean basins, possibly as a result of the nonlinear barotropization of the geostrophic eddy field. Naveira Garabato et al. (2004) have suggested that the generation of quasi-steady lee waves by geostrophic flows is an alternative explanation for the enhanced wave activity in the Drake Passage region. This route is explored here with emphasis on the amount of diabatic mixing that can be supported by internal wave radiation.

The main result of this paper, supported by both idealized simulations and linear theory, is that internal waves generated by geostrophic flows can support enhanced abyssal mixing. To relate this work to observations in the SO, parameters characteristic of flows and topography in Drake Passage are chosen. However, the approach is highly idealized and is used to explain qualitative aspects of the radiation and dissipation problem only. A quantitative comparison using more realistic model configurations is the focus of a companion paper (Nikurashin and Ferrari 2010, hereafter NF) where we show, using multibeam topography, velocity, and stratification data available for the SO, that the parameter space described here is relevant to the SO. The amount of energy dissipation diagnosed from the idealized simulations is of the same order of magnitude as dissipation inferred from observations and strongly suggests that the geostrophic flow–topography interaction is an important process for abyssal mixing in the SO.

This paper is organized as follows: in section 2, the linear theory for topographic internal wave generation is reviewed and the nondimensional parameters that characterize the properties of internal wave generation in the ocean are introduced; in section 3, the theory for the generation of internal waves by a geostrophic flow is presented, with a major focus on the analysis of the feedback that drives IOs and leads to wave breaking and mixing; in section 4, the setup of the numerical experiment used to test the theory is described; in section 5, the numerical simulations are analyzed and compared to linear theory predictions; and in section 6, the conclusions are presented.

#### 2. Topographic wave theory

The goal of this study is to describe the generation of internal waves by geostrophic flows over topography in the abyssal ocean. The problem configuration is very idealized and limited to 2D to focus on the essential physics. [Nikurashin (2009) shows that our results apply to 3D as well.] The numerical simulations described in this study show that the radiation and breaking of internal waves trigger vigorous IOs at the ocean bottom. The development of IOs makes the problem time dependent, unlike the classical steady lee-wave problem. More importantly, vertical shear associated with IOs promotes enhanced internal wave breaking and dissipation. In this section, we start by reviewing the theory of topographic wave generation by both steady and oscillatory flows and the resulting feedback on the largescale flow. Then, we go through a systematic analysis of the physics that triggers IOs and show that they are an inevitable consequence of the interaction between geostrophic flows and topography. A reader interested primarily in the analysis of the numerical simulations and their implications for turbulent mixing can skip to section 4.

#### a. Generation of waves by topography

In the classical lee-wave problem (Long 1953; Bretherton 1969a; Gill 1982), a constant flow in a stratified fluid over a variable bottom topography generates steady, upward radiating internal waves. The essential physics is captured by a 2D, horizontally periodic, and vertically unbounded domain with a sinusoidal bottom topography given by  $h(x) = h_0 \cos kx$ . For small amplitude waves and a uniform mean flow  $U_0$  and stratification N, the linearized Navier-Stokes equations are

$$\begin{split} U_0 \mathbf{u}_x + f \hat{\mathbf{z}} &\times \mathbf{u} = -\nabla p + b \hat{\mathbf{z}}, \\ U_0 b_x + N^2 w &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \\ w|_{z=0} &= U_0 h_x, \end{split}$$

where  $\mathbf{u} = (u, v, w)$ , *b*, and *p* are the wave velocity, buoyancy, and pressure fields, respectively, *f* is the Coriolis frequency, and  $\mathbf{\nabla} = (\partial_x, \partial_z)$ . If  $U_0$  is constant, then solutions to the linear problem are in the form of monochromatic, stationary lee waves with horizontal wavenumber *k* and vertical wavenumber *m*,

$$m = k \sqrt{\frac{N^2 - U_0^2 k^2}{U_0^2 k^2 - f^2}}.$$
 (1)

Lee waves can radiate upward only if  $f < U_0 k < N$  and m is real. If k is in the radiative range, lee waves transport energy and momentum upward at the following rates (Eliassen and Palm 1961; Bretherton 1969b):

$$\overline{pw}^{x} = \frac{1}{2}\rho_{0}U_{0}^{3}h_{0}^{2}km\left(1 - \frac{f^{2}}{U_{0}^{2}k^{2}}\right) > 0,$$
  
$$\overline{uw}^{x} = -\frac{1}{2}\rho_{0}U_{0}^{2}h_{0}^{2}km < 0,$$
 (2)

which are independent of height  $\overline{(\cdot)}^x$  denotes a spatial average over one topographic wavelength]. A momentum flux divergence appears only if the waves break and deposit their negative momentum. In a nonrotating reference frame, the resulting momentum deposition acts to slow down the mean flow and reduce the mean kinetic energy (MKE). In a rotating reference frame, the force generated by the deposition of wave momentum slows down the mean flow and triggers near-inertial oscillations and higher-frequency internal waves, much like wind stress does at the ocean surface (Vadas and Fritts 2001; Lott 2003).

If the mean flow oscillates at a frequency  $\omega_0$ , waves are radiated from the topography both at the fundamental frequency  $\omega_0$  and all its superharmonics (Bell 1975a,b) generating time-mean energy and momentum fluxes:

$$\overline{pw}^{x,t} = \frac{1}{2}\rho_0 h_0^2 k^{-2} \sum_{n=-\infty}^{\infty} \omega_n m_n (\omega_n^2 - f^2) J_n^2 \left(\frac{U_0 k}{\omega_0}\right),$$
$$\overline{uw}^{x,t} = 0,$$
(3)

where  $\omega_n = n\omega_0$  is the *n*th harmonic of  $\omega_0$ ,  $m_n = k(N^2 - \omega_n^2)^{1/2}(\omega_n^2 - f^2)^{1/2}$  is its vertical wavenumber, and  $J_n$  is the *n*th Bessel function of the first kind. The operator  $\overline{(\cdot)}^{x,t}$  denotes a time average over one period of the mean flow, in addition to the spatial average over one topographic wavelength. The mean vertical momentum flux is zero because waves transport an equal amount of vertical momentum both upward and downward during one period of the oscillations.

If water parcels travel a short distance over the topographic bump during one period of oscillation  $U_0k/\omega_0 \ll 1$ , the wave energy and momentum are radiated mostly at the fundamental frequency. However, for large  $U_0k/\omega_0$ , the particle excursion becomes greater than the scale of topography and internal waves are generated in a form of quasi-steady lee waves, which are reinforced at successive cycles by the harmonics of the fundamental frequency, making the wave field multichromatic (Bell 1975a; Garrett and Kunze 2007).

Although the steady component of the momentum flux associated with oscillatory flows is zero, it has time-dependent components  $\overline{uw}^x = \overline{uw}^x|_{\omega=\pm\omega_0} + \overline{uw}^x|_{\omega=\pm2\omega_0} + \cdots$ , where the terms on the right-hand side are the flux components oscillating at harmonics of  $\omega_0$ . When time-dependent waves break and deposit their momentum, they can result in a time-dependent forcing on the zonally averaged flow.

In the SO, abyssal flows are dominated by geostrophic flows, inertial oscillations, and tides (Nowlin et al. 1986). All these motions can radiate internal waves through interaction with bottom topography. Whereas geostrophic flows can be regarded as quasi-steady on the internal wave time scale, inertial oscillations and tides are oscillatory flows. Combining the results for wave radiation by steady and oscillatory flows, the total internal wave momentum flux can be written as a superposition of steady, inertial f, and tidal  $\omega_T$ , waves, their higher harmonics and their linear combinations, so that

$$\overline{uw}^{x} = \overline{uw}^{x}\big|_{\omega=0} + \overline{uw}^{x}\big|_{\omega=\pm f} + \overline{uw}^{x}\big|_{\omega=\pm\omega_{T}} + \cdots$$
(4)

The steady component of the momentum flux describes the time-mean internal wave radiation. The timedependent momentum-flux components force a fast time response in the large-scale flow at their corresponding frequencies. The components at a frequency other than



FIG. 1. Definition sketch illustrating the interaction of a geostrophic flow  $U_G$  and an IO  $(U_I, f)$  with periodic bottom topography  $(k_T, h_T)$  in stably stratified fluid. Dashed gray lines are the phase lines of the waves and the dashed black line is the trajectory of a particle, illustrating its excursion during one period of oscillation.

*f* have little effect on the evolution of the mean flow at subinertial time scales. The momentum flux component oscillating at frequency  $\pm f$  is different, because *f* is the natural oscillation frequency in the rotating system and a forcing at *f* can drive a resonant response. This resonance is crucial because it modifies the subsequent wave generation and breaking. This will be explored in detail in section 3.

# b. The fundamental nondimensional parameters

A formal derivation of internal wave generation by a mean flow and the associated inertial response are the topic of section 3. Here, we introduce the nondimensional parameters that characterize the problem. Let us consider the situation sketched in Fig. 1. A zonal flow, composed of the superposition of a steady geostrophic flow of amplitude  $U_G$  and an oscillatory flow of frequency f and amplitude  $U_I$ , impinges on a sinusoidal topography of amplitude  $h_T$  and wavenumber  $k_T$ . The geostrophic and inertial flows are depth independent, while initially density has a constant stratification N. These parameters can be collapsed into four nondimensional numbers characterizing the different dynamical regimes that can develop in the problem.

The first two nondimensional numbers determine whether the topographic waves can radiate or remain trapped above topography. Geostrophic flows can radiate stationary lee waves flowing over topographic features with scales between  $U_G/f$  and  $U_G/N$  [see Eq. (1)]. Hence, wave radiation is possible only if the nondimensional parameter  $\chi = U_G k_T/N$ , the ratio of the intrinsic lee-wave frequency and buoyancy frequency, lies in the range  $f/N < \chi < 1$ , where the Prandtl ratio f/N is the second nondimensional parameter.

The third and dynamically most significant nondimensional number is the steepness parameter that controls the degree of nonlinearity of the waves. It is defined as the ratio of the topographic slope  $k_T h_T$  to the slope of the internal wave phase lines  $k_T/m$ , where *m* is the vertical wavenumber of the waves,

$$\epsilon = mh_T \approx \frac{Nh_T}{U_G},\tag{5}$$

where  $m \approx N/U_G$  for radiating lee waves such that  $f/N \ll \chi \ll 1$  [see Eq. (1)]. The time dependence in the bottom velocity results in the generation of waves with different vertical structure and breaks this simple relationship. However, in our problem, the time-dependent component of the flow  $U_I$  is generated by  $U_G$  and hence (5) remains a useful parameter to characterize different dynamical regimes. The steepness parameter  $\epsilon$  is used to distinguish different topography regimes: subcritical  $\epsilon \ll 1$ , critical  $\epsilon \sim 1$ , and supercritical  $\epsilon \gg 1$ . In the subcritical regime, the waves are essentially linear. In the critical and supercritical regimes, nonlinearity becomes important and results in a low-level wave breaking and flow-blocking effects.

The frequency of the waves radiated by a timedependent flow is controlled by the fourth nondimensional number, the excursion parameter  $\beta = U_I k_T / f$ . This number compares the amplitude of a particle excursion during one oscillation  $U_I f^{-1}$  to the horizontal scale of the topographic bumps  $k_T^{-1}$ . For  $\beta \ll 1$ , the particle excursion is less than the scale of topography and the waves radiate mainly at the fundamental frequency f, Doppler shifted by  $U_G k_T$ . For  $\beta \sim 1$ , the particle excursion is comparable with the scale of topography and superharmonics of the fundamental frequency are radiated making the wave field multichromatic. For  $\beta \gg 1$ , one recovers the quasi-steady lee-wave regime.

It is useful to estimate the nondimensional numbers for the flows observed in the Drake Passage region, used here as a prototype situation for the idealized problem. A more thorough comparison is given in a companion paper (NF). The lowered acoustic Doppler current profiler (LADCP) and CTD data (Naveira Garabato et al. 2002, 2003) show that in the core of the Antarctic Circumpolar Current (ACC) geostrophic eddy velocities at the bottom are typically  $U_G \sim 0.1 \text{ m s}^{-1}$ , the stratification is  $N \sim 10^{-3} \text{ s}^{-1}$ , and the Coriolis frequency is  $f \approx$  $10^{-4} \text{ s}^{-1}$ . Linear wave theory (Bell 1975a,b) suggests that for these parameters the lee-wave energy flux is largest for topographic wavenumbers close to  $k_T \approx 2\pi/2 \text{ km}^{-1}$ . Multibeam data collected by the British Antarctic Survey for Drake Passage show that the typical height of topographic hills at these scales is close to  $h_T = 60$  m; with these values we have  $\chi \approx 0.3$ ,  $f/N \approx 0.1$ , and  $\epsilon \approx 0.6$ . Both numerical simulations and theory described below suggest that IOs reach the same amplitude as the mean geostrophic flow  $U_I \approx U_G = 0.1$  m s<sup>-1</sup>, corresponding to  $\beta \approx 3$ . With these parameters, radiation of internal waves is close to critical and advective effects are large enough to generate harmonics beyond the fundamental frequency f. This is confirmed in the more extensive analysis of multichromatic topography and realistic mean flows presented in NF. Here, the parameters are given to orient the reader with what a regime characteristic of the Drake Passage region is; however, we will study radiation for a wide range of  $\epsilon$ , f/N, and  $\beta$ .

A number of analytical models of topographic internal wave generation have been developed for both internal tides and lee waves. Ocean models have been mostly used to predict the conversion of energy from barotropic into internal tides, whereas the atmospheric literature focused on the steady lee-wave problem. The linear approach for the ocean was developed by Bell (1975a,b), who considered a barotropic current flowing over topography in a vertically unbounded ocean with uniform stratification. He restricted the analysis to small topographic slopes ( $\epsilon \ll 1$ ) so that topography was subcritical everywhere and the bottom boundary condition could be linearized. With this simplification, solutions were found for arbitrary topography.

Bell's assumption of infinite depth has been the focus of recent work on tidal generation (e.g., Llewellyn Smith and Young 2002), but it is not a major issue for internal waves generated by geostrophic flow and inertial oscillation. Unlike internal tides, these waves are radiated with vertical scales much shorter than the ocean depth, and their generation is not directly affected by the surface boundary condition. The assumption of subcritical topography is more questionable because internal waves are generated by small-scale topographic features, which can be quite steep. We will therefore compare the results of linear theory, valid for small  $\epsilon$ , with numerical simulations in the finite  $\epsilon$  limit. Finally, we follow Bell's approach and make no assumption of small  $\beta$  [as generally done in recent tidal studies, e.g., Llewellyn Smith and Young (2002) and Balmforth et al. (2002)] because IOs can be as large as the geostrophic flow—that is,  $\beta = O(1)$ .

# 3. Multiscale analysis of topographic wave-mean flow interaction

The goal of this section is to derive a set of equations to study the feedback of topographic waves on the flow that generated them. To make analytical progress, we make a judicious choice of scales that allows a clean separation between the equations that describe the wave generation by the large-scale flow and those for the feedback on the large-scale flow. The assumed scale separations are only marginally realized in the real ocean and in the numerical simulations described in this study; however, the goal here is to unfold the underlying physics, not necessarily to make accurate quantitative predictions. We idealize the ocean as a Boussinesq, rotating, and stably stratified fluid governed by

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_H)\mathbf{u} + w\mathbf{u}_z + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla_H p + \mathcal{D}_m(\mathbf{u}), \quad (6)$$

$$w_t + (\mathbf{u} \cdot \nabla_H)w + ww_z = -p_z + b + \mathcal{D}_m(w),$$
(7)

$$b_t + (\mathbf{u} \cdot \nabla_H)b + wb_z + wN^2 = \mathcal{D}_b(b), \text{ and } (8)$$

$$\boldsymbol{\nabla}_{H} \cdot \mathbf{u} + \boldsymbol{w}_{\tau} = 0, \tag{9}$$

where  $\mathbf{u} = (u, v)$  and w are, respectively, the horizontal and vertical velocities, p is pressure,  $b = -g(\rho - \rho_0)/\rho_0$  is buoyancy, f is the Coriolis frequency, N is the buoyancy frequency, and  $\rho_0$  is a reference density. Dissipation of buoyancy and momentum,  $\mathcal{D}_m$  and  $\mathcal{D}_b$ , are assumed to be linear operators as described later.

The flow is assumed to be periodic in the horizontal. Top and bottom boundary conditions are of zero velocity normal to topography and vanishing vertical velocity as  $z \rightarrow \infty$ ,

$$w|_{z=h(x)} = \mathbf{u} \cdot \nabla_H h(x) \text{ and } w|_{z\to\infty} = 0.$$
 (10)

First, we nondimensionalize both the governing equations and the boundary conditions, and then we expand the solution in the small steepness parameter  $\epsilon = Nh_T/U_G$ . The steepness parameter characterizes the nonlinearity in the bottom boundary condition. We assume that the nonlinearity in the momentum equation is of  $O(\epsilon)$ , consistent with lee-wave scaling (see appendix A). At leading order, the flow  $\mathbf{u}^{(0)}$  is composed of a geostrophic component  $\mathbf{u}^G$  and an IO  $\mathbf{u}^I = U_I(\cos f(t - t_0), -\sin f(t - t_0))$ , where  $U_I$  and  $ft_0$  are the amplitude and phase of IO, respectively. The radiating waves are assumed to be of small amplitude  $\mathbf{u}^{(1)}$ . The full expansion of the velocity field takes the form

$$\mathbf{u} = \mathbf{u}^G + \mathbf{u}^I + \epsilon \mathbf{u}^{(1)} + \epsilon^2 \mathbf{u}^{(2)} + \epsilon^3 \mathbf{u}^{(3)} + \cdots, \quad (11)$$

and scales are expanded as  $t \to t + e^{-1}T^{(1)} + \cdots$  and  $\mathbf{x} \to \mathbf{x} + e^{-1}\mathbf{X}^{(1)} + \cdots$ , where  $(t, \mathbf{x})$  are the scales of the waves and  $[T^{(n)}, \mathbf{X}^{(n)}]$  are slower scales of the problem. We assume that the geostrophic flow varies on slow subinertial scales  $\mathbf{u}^G = \mathbf{u}^G[T^{(4)}, \mathbf{X}^{(4)}]$  so that the evolution of  $\mathbf{u}^{G}$  enters at higher orders than any of the dynamics described here. This is tantamount to assuming that the Rossby number associated with  $\mathbf{u}^G$  is  $O(\epsilon^4)$ . The amplitude of the IO also varies on slow scales  $U_I = U_I[T^{(3)}]$ ,  $X^{(4)}, Y^{(4)}, Z^{(1)}$ , that is, we assume that the IO varies on the same scale as  $\mathbf{u}^{G}$  in the horizontal, but has a smaller vertical scale; the numerical simulations described in this study confirm that  $U_I$  varies in the vertical on scales of O(1) km, smaller than the vertical scale of geostrophic eddies, which is of O(3-4) km in the abyss. The time scale of the evolution of  $U_I$  is set by the dynamics, where  $U_I$  is forced by the vertical divergence on scale  $Z^{(1)}$  of the wave momentum flux divergence  $\partial_{z^{(1)}} \overline{\mathbf{u}^{(1)} w^{(1)}} = O(\epsilon^3)$ . Once these judicious choices are made for the leadingorder flow, the dynamical equations determine the evolution of the problem.

At leading-order  $O(\epsilon^0)$ , we obtain a set of equations describing both the geostrophic flow

$$f\hat{\mathbf{z}} \times \mathbf{u}^{G} = -\nabla_{\mathbf{X}^{(4)}} p^{G},$$
  
$$0 = -p_{Z^{(4)}}^{G} + b^{G},$$
  
$$w^{G} = 0$$
(12)

and the IO

$$\mathbf{u}_t^I + f\hat{\mathbf{z}} \times \mathbf{u}^I = 0, \quad w^I = 0.$$
(13)

Dissipation on the large scale of the leading-order flow is assumed to be weak and does not appear at this order. This corresponds to expanding in a small damping parameter (see appendix A).

At the order  $O(\epsilon^1)$ , we get the set of equations and boundary conditions governing the evolution of internal waves generated by the leading-order motions interacting with topography:

$$\mathbf{u}_{t}^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_{\mathbf{x}} \mathbf{u}^{(1)} + f\hat{\mathbf{z}} \times \mathbf{u}^{(1)} = -\nabla_{\mathbf{x}} p^{(1)} + \mathcal{D}_{m}^{(1)}[\mathbf{u}^{(1)}],$$
(14)

$$0 = -p_z^{(1)} + b^{(1)}, \qquad (15)$$

$$b_t^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_{\mathbf{x}} b^{(1)} + N^2 w^{(1)} = \mathcal{D}_b^{(1)}[b^{(1)}],$$
(16)

$$\nabla_{\mathbf{x}} \cdot \mathbf{u}^{(1)} + w_z^{(1)} = 0, \text{ and } (17)$$

$$w^{(1)}|_{z=0} = \mathbf{u}^{(0)} \cdot \nabla_{\mathbf{x}} h, \quad w^{(1)}|_{z\to\infty} = 0.$$
 (18)

The  $O(\epsilon^2)$  equations (see appendix A) describe  $O(\epsilon^2)$ internal waves generated by the lower-order motions flowing over topography, as well as by direct forcing from  $O(\epsilon^1)$  internal waves through nonlinear terms. These waves do not drive any flux averaged over large scales and hence do not feedback on the mean flow. The wave–mean flow interaction appears at  $O(\epsilon^3)$ .

The coarse-grained averaged momentum equation at  $O(\epsilon^3)$  is

$$\overline{\mathbf{u}_{t}^{(3)}} + f\hat{\mathbf{z}} \times \overline{\mathbf{u}^{(3)}} = -\mathbf{u}_{T^{(3)}}^{I} - \partial_{Z^{(1)}}\overline{\mathbf{u}^{(1)}w^{(1)}} + \mathcal{D}_{m}^{(3)}[\overline{\mathbf{u}^{(3)}}],$$
(19)

where the overbar is an average over small spatial scales **x**. At this order, there is a fast time evolution of the  $O(\epsilon^3)$  IO combined with a slow time evolution of the leadingorder IO  $\mathbf{u}^I$ . The IOs are forced by the  $O(\epsilon^3)$  internal wave momentum flux divergence. The evolution of the geostrophic flow occurs on a slower subinertial time scale  $T^{(4)} \gg T^{(3)}$ .

# *a.* $O(\epsilon^1)$ solution: Wave generation

In this section, we solve for the generation of  $O(\epsilon^1)$  internal waves in 2D for a monochromatic bottom topography  $h(x) = h_T \cos(k_T x)$ . The leading-order flow, which is governed by Eqs. (12) and (13), can be written as

$$u^{(0)}(t) = U_G + U_1 \cos f(t - t_0),$$
  

$$v^{(0)}(t) = -U_1 \sin f(t - t_0),$$
(20)

where  $U_G$  is a zonal geostrophic flow varying on the slow variables  $[T^{(4)}, \mathbf{X}^{(4)}]$ , and  $U_I$  and  $f_{t_0}$  are the amplitude and phase of IO, respectively, varying on the slow variables  $[T^{(3)}, X^{(4)}, Y^{(4)}, Z^{(1)}]$ .

To make analytical progress we represent dissipation as a linear damping,

$$D_m^{(n)}(\mathbf{u}^{(n)}) = -\lambda \mathbf{u}^{(n)},$$
  
$$D_b^{(n)}(b^{(n)}) = -\lambda b^{(n)},$$
 (21)

where  $\lambda$  is the dissipation rate. The dissipation of momentum and buoyancy is retained here, unlike in Bell (1975a,b), to represent in a crude way the effect of smallscale turbulence that damps radiating linear waves.

Analytical solutions to (14)–(18) are found by switching to a reference frame moving with the zonally averaged flow,  $\xi = x - \int_{t_0}^t u^{(0)}(t') dt'$ . The problem can be reduced to a single equation for the vertical velocity  $w^{(1)}$ :

$$(\partial_{tt} + 2\lambda \partial_t + \lambda^2) w_{zz}^{(1)} + N^2 w_{\xi\xi}^{(1)} + f^2 w_{zz}^{(1)} = 0, \qquad (22)$$

$$w^{(1)}|_{z=0} = u^{(0)}h_{\xi}.$$
 (23)

Solutions matching a periodic topography are found by expanding variables into Fourier modes in the  $\xi$ -coordinate frame. The solution to (22) that satisfies the boundary condition (23) is

$$w^{(1)}(t, \xi, z) = -h_T \sum_{n=-\infty}^{\infty} \sigma_n J_n(\beta) \operatorname{Im}(e^{i\theta_n}),$$
  
$$\theta_n = k_T \xi + m_n z + \sigma_n (t - t_0), \qquad (24)$$

where  $\sigma_n = nf + U_G k_T$  is the intrinsic frequency of the *n*th harmonic of the inertial frequency Doppler shifted by  $U_G$ ;  $m_n$  is a complex number with the real and imaginary parts representing, respectively, the wave vertical wavenumber and an inverse decay scale resulting from damping,

$$m_n^2 = k_T^2 \frac{N^2}{(\sigma_n - i\lambda)^2 - f^2};$$
 (25)

 $\beta$  is the excursion parameter, and  $J_n(\beta)$  are Bessel functions of the first kind. For finite  $\beta$ , all harmonics *n* have comparable magnitude (only at small  $\beta$  most of the energy is in the fundamental frequency *f*). The fact that the internal wave field is a superposition of inertial frequency harmonics has important implications for the wave–mean flow feedback mechanism discussed in the next section.

All other dynamical variables can be reconstructed from  $w^{(1)}$ . The rate of the bottom energy conversion from the geostrophic flow and the IO averaged zonally and over an inertial period can be written as

$$\overline{p^{(1)}w^{(1)}} = \frac{1}{2} \sum_{n=-\infty}^{\infty} h_T^2 \sigma_n^3 k_T^{-1} J_n^2(\beta) \operatorname{Re}\left[\frac{m_n}{k_T} \frac{(\sigma_n - i\lambda)^2 - f^2}{\sigma_n(\sigma_n - i\lambda)}\right].$$
(26)

This expression reduces to the lee-wave expression in the limit of  $\beta = 0$  (no inertial oscillation in the leading-order flow) and to the expression for internal tide conversion, if the tidal frequency is used instead of the inertial frequency (Bell 1975a,b).

# b. $O(\epsilon^3)$ solution: Feedback of waves on the large-scale flow

We now consider the feedback of internal waves on the leading order flow. Averaging the  $O(\epsilon^3)$  equations over the short wave scales gave us (19), which describes the slow time evolution of the  $O(\epsilon^0)$  IO, driven by the divergence of momentum fluxes associated with  $O(\epsilon^1)$ waves. Introducing the complex velocity  $\mathcal{V} = u + iv$ , and representing a linear damping operator as already done for  $\mathcal{D}^{(1)}$ , Eq. (19) becomes

$$\overline{\mathcal{V}_t^{(3)}} + i f \overline{\mathcal{V}^{(3)}} = -\mathcal{V}_{T^{(3)}}^I - \partial_{Z^{(1)}} \overline{\mathcal{V}^{(1)} w^{(1)}} - \lambda \overline{\mathcal{V}^{(3)}}, \quad (27)$$

where  $\mathcal{V}_{T^{(3)}}^{I}$  is the evolution of <u>the leading-order IO</u> on the slow time scale  $T^{(3)}$ , and  $\partial_{Z^{(1)}} \mathcal{V}^{(1)} w^{(1)}$  is the divergence

of the internal wave momentum flux. Although this equation is valid at any vertical level, in this section we focus on the feedback at z = 0, where the leading-order flow and the  $O(\epsilon^1)$  internal waves are coupled through the bottom boundary condition.

First, we want to illustrate the instability that generates the IOs. To do so, we assume that the amplitude of the IO is much smaller than the geostrophic flow at  $t = t_0$ (even though it is still of leading order)—that is,  $U_I \ll$  $U_G$ . This implies  $\beta = U_I k_T / f \ll 1$ , but also  $U_G k_T / f > 1$ , to be in the radiative range. The  $O(\epsilon^1)$  vertical velocity (24) in the small  $\beta$  limit takes the following form:

$$w^{(1)} = -h_T \sigma_0 \text{Im}(e^{i\theta_0}) + \frac{1}{2}\beta h_T \sigma_{\pm 1} \text{Im}(e^{i\theta_{\pm 1}}) + O(\beta^2).$$
  
$$\theta_n = k_T \xi + m_n z + \sigma_n (t - t_0).$$
(28)

The first term represents a wave generated by the geostrophic flow with intrinsic frequency  $\sigma_0 = U_G k_T$  in the moving reference frame; this reduces to a lee wave in the absence of an IO. The second term is generated by the IO, and it is the superposition of two waves with intrinsic frequencies  $\sigma_{\pm 1} = \pm f + U_G k_T$ . The amplitude of the primary wave is proportional to  $U_G$  and much larger than the amplitudes of the oscillatory waves, which are proportional to  $U_I$ .

The small- $\beta$  expression for the wave momentum flux divergence at z = 0 is computed from  $w^{(1)}$  and the corresponding expression for  $\mathcal{V}^{(1)}$ 

$$\partial_{Z^{(1)}} \overline{\mathcal{V}^{(1)} w^{(1)}} = A - \beta [Be^{-if(t-t_0)} + Ce^{if(t-t_0)}] + O(\beta^2),$$
(29)

where *A*, *B*, and *C* are constants that depend on external parameters of the problem, as shown in appendix B. Apart from a dependence on fixed external parameters, the wave flux divergence is proportional to the slowly varying amplitude  $U_I$ , through  $\beta$ , and the phase  $t_0$  of the IO. In the absence of an IO,  $\partial_{Z^{(1)}} \mathcal{V}^{(1)} w^{(1)}$  is constant and no instability can develop. The presence of an IO in the bottom flow forces a component of  $\partial_{Z^{(1)}} \mathcal{V}^{(1)} w^{(1)}$  oscillating at *f*, which drives a resonant response. This flux is formed by the product of the wave with  $\sigma_0 = U_G k_T$  and each of the two oscillatory waves with  $\sigma_{\pm 1} = \pm f + U_G k_T$ .

Substituting the expression for the momentum flux divergence (29) into (27) we have

$$\overline{\mathcal{V}_{t}^{(3)}} + if\overline{\mathcal{V}^{(3)}} = -A - (\dot{U}_{I} + ifU_{I}\dot{t}_{0} - U_{I}k_{T}f^{-1}B)e^{-if(t-t_{0})} + U_{I}k_{T}f^{-1}Ce^{if(t-t_{0})} - \lambda\overline{\mathcal{V}^{(3)}},$$
(30)

where  $\mathcal{V}^{I} = U_{I}e^{-if(t-t_{0})}$ . The solvability condition for the evolution of  $U_{I}$  and  $t_{0}$  on the slow time scale  $T^{(3)}$  is obtained by setting the secular terms to zero. The solutions to the problem are

$$U_{I}(T^{(3)}) = U_{I}(0)e^{k_{T}f^{-1}\operatorname{Re}(B)T^{(3)}} \text{ and}$$
  
$$t_{0}(T^{(3)}) = k_{T}f^{-2}\operatorname{Im}(B)T^{(3)};$$
(31)

that is,  $U_I$  grows exponentially on the slow time scale  $T^{(3)}$  as long as  $\Gamma = k_T \operatorname{Re}(B)/f > 0$ .

In the limit of hydrostatic  $(U_G k_T \ll N)$ , superinertial  $(U_G k_T \gg f)$ , and weakly damped  $(U_G k_T \gg \lambda)$  waves, the expression for the growth rate of the inertial flow becomes

$$\Gamma \simeq \frac{1}{2}\lambda\epsilon^2 \left(1 + 4\frac{f^2}{U_G^2 k_T^2} - 3\frac{\lambda^2}{U_G^2 k_T^2}\right).$$
 (32)

This expression illustrates the fundamental physics at play. The instability develops only in the presence of damping  $\lambda$ , otherwise the waves cannot deposit momentum and feedback on the large-scale flow, a limit known as the nonacceleration conditions in the atmospheric literature (Eliassen and Palm 1961; Andrews and McIntyre 1976). Furthermore, the instability is proportional to  $\epsilon^2$ , that is, it is more rapid for larger-amplitude topography (implying large-amplitude waves). When the damping exceeds a critical value greater than  $2/3^{(1/2)f}$ , the instability is suppressed.

The vanishing of the instability for  $\lambda = 0$  deserves more explanation. For  $\lambda = 0$ , the amplitude of the wave momentum flux is vertically uniform. The momentum flux and its vertical divergence are in quadrature with the IO at every level, and their net work over a period is zero. When  $\lambda > 0$ , the amplitude of the wave momentum fluxes decays with height, resulting in wave momentum deposition and a phase shift between the IO and the wave momentum flux divergence. When slightly out of phase, the wave momentum fluxes work to accelerate the IO. For large  $\lambda$ , the phase shift becomes so large that wave momentum deposition can start to work against IOs.

The multiscale expansion also predicts the saturation of the instability. While at small  $\beta$ , the magnitude of the wave momentum fluxes increases linearly with  $\beta$ , causing exponential growth of  $U_I$ ; at finite  $\beta$ , it is proportional to  $J_n(\beta)$  (see appendix B) and can decrease with  $\beta$ . At a certain finite value of  $\beta$  when the magnitude of the resonant flux component vanishes, the instability is suppressed. The saturation of the instability at finite  $\beta$  is demonstrated in section 5, using the full expression for the wave momentum fluxes derived in appendix B in comparison with results from the numerical simulations.

#### 4. Numerical model setup

We use the Massachusetts Institute of Technology general circulation model (MITgcm), which solves the nonhydrostatic, nonlinear primitive equations using a finite-volume formulation (Marshall et al. 1997). By running the model in the nonhydrostatic form, hydraulic jumps and Kelvin–Helmholtz instabilities, which develop in our problem, are explicitly resolved without the need for parameterizations. The MITgcm has been used for studies of wave radiation and breaking before (e.g., Khatiwala 2003; Legg and Huijts 2006).

The domain used in the simulations is 2D, horizontally periodic with a uniform resolution of  $\Delta x = 12.5$  m in the horizontal and variable resolution in the vertical. The vertical grid spacing is set to  $\Delta z = 5$  m in the bottom 2 km, and it is gradually stretched to  $\Delta z = 300$  m in the region above. The domain size is  $L \times H = 2 \text{ km} \times 7 \text{ km}$ . To absorb upward-propagating internal waves, a sponge layer is applied between 2 km above the topography and the top boundary, where buoyancy and momentum are damped with a time scale of 4 h. A uniform stratification of  $N = 10^{-3} \text{ s}^{-1}$  and a Coriolis frequency of  $f = 10^{-4} \text{ s}^{-1}$ are used. Horizontal and vertical viscosity and diffusivity are set to  $10^{-2}$  m<sup>2</sup> s<sup>-1</sup> and  $10^{-3}$  m<sup>2</sup> s<sup>-1</sup> respectively; experiments with higher resolution and lower values of viscosity and diffusivity,  $10^{-3}$  m<sup>2</sup> s<sup>-1</sup> and  $10^{-4}$  m<sup>2</sup> s<sup>-1</sup>, respectively, show quantitatively similar results.

Bottom topography has the form  $h(x) = h_T \cos k_T x$ with a wavenumber  $k_T = 2\pi/2 \text{ km}^{-1}$  and an amplitude  $h_T$  varying from 10 to 80 m. A depth-independent mean geostrophic flow  $U_G = 0.1 \text{ m s}^{-1}$  is forced by adding a body force  $fU_G$  to the meridional momentum equation. This body force maintains a zonal vertically uniform flow in geostrophic balance and is analogous to a tilt of the thermocline in a two-layer model. Here, freeslip bottom boundary conditions are imposed. Other boundary conditions are considered in section 5c.

All experiments are initiated from a state of rest. Then, the velocity and temperature fields are slowly relaxed to the desired basic state for a 24-h time period, with a relaxation time scale of 3 h. A gradual increase of the flow to the basic-state value is necessary to let the flow adjust to the bottom topography and avoid spurious initial transient effects. After the first day, the relaxation term is removed and the system is integrated for nine more days; at longer times one should include the evolution of the geostrophic flow.

#### 5. Analysis of numerical simulations

In this section, we use the theory described above to interpret the results of the numerical simulations. We decompose the model solution into a zonally averaged



FIG. 2. (top) Time evolution of zonally averaged velocities (m s<sup>-1</sup>) from a simulation with  $\epsilon = 0.4$ . Only a perturbation velocity in addition to an externally prescribed 0.1 m s<sup>-1</sup> zonal flow is shown. (bottom) The corresponding zonally averaged meridional velocity component.

flow and deviations from the zonally averaged flow (the waves). First, we discuss the evolution of the zonally averaged flow and demonstrate that the growth of IOs in the simulations is consistent with the resonant feedback mechanism described in section 3. Next, we compare the internal wave fluxes from simulations with the theoretical predictions. Finally, we discuss the energy budget of wave radiation, breaking, and dissipation, and show the relevance of our results for turbulent dissipation estimates in the real ocean.

# a. IOs

The time evolution of the zonally averaged flow from a simulation with  $\epsilon = Nh_T/U_G = 0.4$  is shown in Fig. 2. The problem becomes time dependent at  $\epsilon = 0.4$ , and this simulation displays the essential physics of interest without the additional complications that arise at large  $\epsilon$ , described later in the study. No IO is imposed in the initial conditions; however, within the first 48 h a strong oscillatory flow with frequency *f* develops and saturates once it reaches the same amplitude of the prescribed geostrophic flow of 0.1 m s<sup>-1</sup>. The IOs extend throughout the whole domain, but they are particularly intense within 700 m of the bottom.

The growth of the IO amplitude at z = 100 m, slightly above the topography, is shown in Fig. 3 for the whole set of simulations with different  $\epsilon$ : the growth rate increases with  $\epsilon$ . For  $\epsilon \le 0.3$ , the growth rate is very slow and the IO amplitude has not reached equilibrium after 10 days of simulation. Simulations with  $\epsilon > 0.3$  have fully developed IOs with an amplitude of about  $0.12 \pm 0.02$  m s<sup>-1</sup>—that is, of the same order as the geostrophic flow.

Although the vertical scale separation between IOs and internal waves assumed in the theory is not well



FIG. 3. Evolution of the bottom value of the IO amplitude (m s<sup>-1</sup>) from different simulations with different  $\epsilon$  values.



FIG. 4. Evolution of the bottom value of the IO amplitude normalized by its initial value  $\log [U_I(t)/U_I(0)]$  from the  $\epsilon = 0.4$  simulation (gray), resonant feedback theory prediction from [(27); solid black], and linear growth rate prediction [(32); dashed black].

satisfied in the simulations, the characteristics of the IOs are consistent with the resonant feedback mechanism described in section 3. Theory predicts that the vertical structure of the IO is set by the divergence of internal wave momentum flux oscillating at frequency f, as per Eq. (27). Consistent with theory, the resonant component of the wave momentum flux is dominated by harmonics  $\sigma_{-1} = -f + U_G k_T$  and  $\sigma_0 = U_G k_T$  and it has a vertical scale of  $2\pi/\text{Re}(m_{-1} - m_0) \sim 1$  km; the phase lines of the inertial oscillations in (t, z) space have a slope very close to  $f/\text{Re}(m_{-1} - m_0)$ .

For  $\epsilon > 0.3$ , waves break above topography and deposit most of their momentum before reaching the sponge layer at 2 km. The IOs are most pronounced in regions where wave breaking occurs. For the reference simulation  $\epsilon = 0.4$ , the momentum flux decays within 700 m of the bottom. In the theoretical model presented in section 3, wave breaking was represented as a lineardamping process. In that model, a damping rate of  $\lambda$  =  $5 \times 10^{-5}$  s<sup>-1</sup> gives  $\overline{u^{(1)}w^{(1)}}$ , with a vertical decay scale of 700 m (as estimated from a least squares fit of the model to the simulation). We discuss what sets this scale below. In Fig. 4, we show that once  $\lambda$  is picked, linear theory captures both the initial growth rate [through Eq. (32)] and the level at which IOs saturate [through integration of Eq. (27)]. Details of the integration of Eq. (27) are given in appendix B. The comparison between theory and simulations is shown only for  $\epsilon = 0.4$ , but the results are as good for all  $\epsilon$  values. This good match builds confidence that the resonant generation is key to the appearance of IOs.



FIG. 5. Profiles of the vertical energy flux (mW m<sup>-2</sup>) from different simulations with different  $\epsilon$  values.

#### b. Wave radiation

We now test the prediction of linear theory for the energy and momentum fluxes radiated by topographic internal waves. We estimate the energy flux  $\overline{p'w'}$  and the momentum flux  $\overline{u'w'}$  from the model solution using deviations from the zonally averaged flow and by averaging the fluxes zonally in space and over several inertial periods in time. These fluxes are the dominant and the most dynamically significant; all terms of the energy budget are estimated for the reference simulation and discussed in section 5c. Figure 5 shows the vertical dependence of the energy flux for various values of  $\epsilon$ . The energy flux at the bottom increases with  $\epsilon$  consistent with the increase in wave amplitude. The bulk of the energy flux decays substantially within less than 1 km of the bottom, as a result of wave breaking and dissipation. Above the breaking level, there is a small residual energy flux that radiates into the ocean interior.

Figure 6 shows the comparison of the bottom values of the energy and momentum fluxes between numerical simulations and the linear theory prediction. To make the theoretical prediction, we use the IO amplitude diagnosed from the simulations in (26)—that is, we set  $U_I = 0.12 \text{ m s}^{-1}$ . The presence of IOs over bottom topography increases the amount of energy radiated by internal waves by about 50% compared to the lee-wave radiation estimate, whereas the wave momentum flux decreases by about 15%, in agreement with the theoretical prediction. However, the importance of IOs is not that they slightly modify the fluxes, but rather they promote wave breaking, as we show in section 5d.

The mean energy flux increases in response to the inclusion of IOs. The energy flux is a positive definite



FIG. 6. (top) Energy and (bottom) momentum fluxes at the bottom as a function of  $\epsilon^2$ : diagnosed from simulations (black squares and circles), predicted from linear lee-wave theory (dashed line), and predicted from linear theory for inertial harmonics (solid line).

quantity, as all waves radiate away from topography by construction, and the upward time-dependent component associated with the IOs adds to the flux component driven by the mean flow. The response of the mean vertical momentum flux to the inclusion of IOs is a bit more complex. It depends on the partitioning between the upward and downward components of the momentum flux. In a steady flow problem, the vertical momentum flux is constant, downward [see Eq. (2)], and it acts against the steady flow. In an oscillatory flow problem, the vertical momentum flux is time dependent; however, waves transport an equal amount of momentum both upward and downward during one period of oscillation, resulting in a zero time-mean momentum flux [Eq. (3)]. In a combination of a steady and an oscillatory flow, the Doppler shift by the steady flow creates an asymmetry between the upward and downward momentum fluxes of the time-dependent waves. It turns out that the downward mean momentum flux driven by the steady flow is reduced by the effect of the timedependent waves.

The characteristics of the radiated waves change as a function of  $\epsilon$  and can be described by three different regimes. The first regime, at small  $\epsilon$  (smaller than 0.3 in our simulations), is characterized by stationary lee-wave generation. IOs do grow in time as a result of the resonant feedback, but the growth rate is small and, over a 10-day period, they do not develop enough to significantly modify the wave generation process. A second regime develops for  $0.3 < \epsilon < 0.7$ , where inertial frequency harmonics are generated. In this  $\epsilon$  range, IOs

grow rapidly and reach an amplitude comparable with that of the geostrophic flow within a few days. These IOs significantly modify the wave generation process by not only increasing the amount of radiated energy but also by making the wave field substantially time dependent and multichromatic. Linear wave theory, modified to account for IOs in the leading-order flow, agrees well with numerical simulations in this  $\epsilon$  range. The last regime  $\epsilon > 0.7$  is characterized by a saturation of the energy flux that ceases to increase with  $\epsilon$ . For large topographic amplitude, the flow in the deep valleys does not have enough kinetic energy to climb back to the top of the hills. This pocket of stagnant fluid acts to reduce the vertical excursion of the fluid flowing over it. As  $\epsilon$ increases, the layer of stagnant fluid thickens so that the layer of fluid flowing over the topography remains constant and the radiated energy saturates.

# c. Energy pathways

In the simulations, a substantial fraction of the energy radiated by gravity waves is dissipated within 1 km of the bottom topography. This layer is characterized by vigorous turbulence resulting from the breaking of internal waves (Fig. 7). To understand the pathways of energy from the prescribed geostrophic flow to wave breaking and turbulent dissipation at small scales, we estimate the mean and wave kinetic energy (EKE) budgets for the layer 1 km above the bottom. The analysis is presented for the  $\epsilon = 0.4$  simulation, but the salient results carry over to larger  $\epsilon$  simulations.

We decompose the model solution into the mean and waves as follows:

$$\mathbf{u} = U_G \hat{\mathbf{i}} + \overline{\mathbf{u}} + \mathbf{u}', \tag{33}$$

where  $U_G$  is the prescribed geostrophic flow,  $\overline{\mathbf{u}}$  is the zonally averaged flow, and  $\mathbf{u}'$  represents wave perturbations. Generally,  $\overline{\mathbf{u}}$  includes both inertial and subinertial flow components. However, the subinertial component is weak because no pressure gradient can develop to balance the zonal subinertial component of  $\overline{\mathbf{u}}$  in a twodimensional zonally periodic domain.

The kinetic energy equation for the zonally averaged flow  $\overline{\mathbf{u}}$ , averaged in time over many inertial periods, takes the form

$$\partial_t \mathbf{MKE} - \nu \partial_{zz} \mathbf{MKE} = -\langle \mathbf{\overline{u}} \cdot \partial_z \overline{w' \mathbf{u}'} \rangle - \varepsilon_{\mathbf{MKE}}, \quad (34)$$

where MKE =  $(\frac{1}{2})\langle \overline{\mathbf{u}} \cdot \overline{\mathbf{u}} \rangle$  is the kinetic energy of the mean flow, and the terms on the right-hand side are the conversion of energy from the zonally averaged flow to wave energy and dissipation of mean energy  $\varepsilon_{\text{MKE}} = \nu \langle |\overline{\mathbf{u}}_z|^2 \rangle$ , respectively. The second term on the left is the transport



FIG. 7. Snapshot of the wave zonal velocity (m s<sup>-1</sup>) from the  $\epsilon = 0.4$  simulation.

of the mean kinetic energy by viscous terms. The overbars and brackets represent spatial and time averages. Similarly, the kinetic energy equation for the wave component  $\mathbf{u}'$  can be written as

$$\partial_{t} \mathbf{E}\mathbf{K}\mathbf{E} - \nu \partial_{zz} \mathbf{E}\mathbf{K}\mathbf{E} = -\partial_{z} \underbrace{\left[ \langle \overline{p'w'} \rangle + U_{G} \langle \overline{w'u'} \rangle + \langle \overline{\mathbf{u}} \cdot \overline{w'u'} \rangle + \frac{1}{2} \langle \overline{w'u'} \cdot \overline{\mathbf{u}'} \rangle \right]}_{(\mathbf{I})} + \underbrace{U_{G} \partial_{z} \langle \overline{w'u'} \rangle}_{(\mathbf{II})} + \underbrace{\langle \overline{\mathbf{u}} \cdot \partial_{z} \overline{w'u'} \rangle}_{(\mathbf{III})} + \underbrace{\langle \overline{b'w'} \rangle}_{(\mathbf{IV})} - \underbrace{\varepsilon_{\mathbf{E}\mathbf{K}\mathbf{E}}}_{(\mathbf{V})},$$
(35)

where the terms on the left-hand side represent the evolution of the wave kinetic energy EKE =  $\frac{1}{2}\langle \mathbf{u}' \cdot \mathbf{u}' \rangle$  and its transport by viscous terms. The terms on the right-hand side are (I) the divergence of the total wave energy flux, (II) the energy exchange with the geostrophic flow  $U_G$ , (III) the energy exchange with the zonally averaged flow  $\mathbf{\bar{u}}$ , (IV) the energy conversion to potential energy, and (V) the wave dissipation rate  $\varepsilon_{\text{EKE}} = \nu \langle |\nabla \mathbf{u}'|^2 \rangle$ . The total wave energy flux (I) includes the pressure work, downward transport of the geostrophic and inertial flow kinetic energies, and energy transport by the triple-correlation term.

The MKE and EKE budgets averaged over the bottom 1 km for the  $\epsilon = 0.4$  simulation are summarized in Tables 1 and 2. Both the mean and wave energy budgets are nearly closed—the residuals are less than 5% of the energy dissipation value. The MKE budget shows that the wave work is largely balanced by viscous dissipation, with only 25% of the work going into accelerating the IOs. To leading order,

$$-\langle \overline{\mathbf{u}} \cdot \partial_z \overline{w' \mathbf{u}'} \rangle \approx \varepsilon_{\mathrm{MKE}}.$$
 (36)

The EKE budget shows that at leading order the energy extracted by the waves from the geostrophic flow  $U_G$  goes in approximately equal parts to supporting IOs and internal wave dissipation,

$$U_G \partial_z \langle \overline{w'u'} \rangle \approx - \langle \overline{\mathbf{u}} \cdot \partial_z \overline{w'\mathbf{u}'} \rangle + \varepsilon_{\text{EKE}}.$$
 (37)

By summing (36) and (37), we have the total energy balance, where energy extracted from the mean geostrophic flow is balanced by dissipation of both waves and IOs:

$$U_G \partial_z \langle \overline{w'u'} \rangle \approx \varepsilon_{\rm MKE} + \varepsilon_{\rm EKE}.$$
 (38)

This balance holds quite well in the bottom 1 km of the water column as shown in Fig. 8. The various energy pathways are summarized in Fig. 9.

TABLE 1. Mean. $\partial_t MKE$  $\nu \partial_{zz} MKE$  $-\langle \overline{\mathbf{u}} \cdot \partial_z \overline{w' \mathbf{u}'} \rangle$  $-\epsilon_{MKE}$ 2.0-0.17.3-5.2

To study the degree to which wave radiation and breaking are sensitive to the choice of the bottom boundary condition, we ran the reference simulation  $\epsilon = 0.4$ , with both free-slip and no-slip boundary conditions. In the latter case, we found that wave dissipation was reduced by about 20%-30%-the simulations were otherwise very similar. It is still possible that boundary conditions have a more profound impact at larger  $\epsilon$  if they lead to a substantial increase in the thickness of the turbulent bottom boundary layer so as to suppress wave generation. However, preliminary tests suggest that these effects become important only for  $\epsilon > 1$ , a regime rarely encountered in open ocean conditions (NF). The question, however, deserves more attention and would require the implementation of a boundary condition capable of resolving the stress in the logarithmic layer.

# d. Wave breaking and dissipation

The numerical simulations show that the generation and dissipation of kinetic energy depends on the vertical divergence of the wave momentum flux. Linear theory gives a good prediction for the wave fluxes at the bottom of the simulation domain. The vertical scale of flux divergence is instead determined by the wave breaking scale. In the simulations described so far, the vertical scale of the wave breaking region is of the order of 700 m and essentially independent of  $\epsilon$ , or more specifically of the topography amplitude  $h_T$  that we varied to change  $\epsilon$ . However, the vertical extent of the breaking region depends on the topographic wavenumber  $k_T$  and the Coriolis frequency f—that is, it depends on the nondimensional parameters  $\chi$ ,  $\beta$ , and *N*/*f*. We find that the wave breaking scale increases with increasing  $\beta$  and *N*/*f* (decreasing *f*), with a weaker dependence on  $\chi$  (through  $k_T$ ).

There are three main pathways from wave generation to wave breaking. First, the radiated waves can have sufficient amplitude to become convectively or shear unstable. This is not the case in our simulations because the linear wave solutions are stable. Second, wave–wave interactions can transfer energy to smaller-scale waves with large shears. This is the classical turbulent picture, where energy is fluxed from larger to smaller scales where instabilities develop and waves break. We can rule out this pathway because the interaction time scale estimated for wave amplitudes observed in our simulations is of the order of a day, which is long enough for waves to radiate up to 2–3 km before they can break (McComas and Muller 1981). Third, modulation by

TABLE 2. Wave.

∂ <sub>t</sub> EKE	<i>v∂</i> <sub>zz</sub> MKE	$-\partial_z F$	$U_G \partial_z \langle \overline{w'u'} \rangle$
-0.2	-0.1	-0.1	13.5
$\langle \overline{\mathbf{u}} \cdot \partial_z \overline{w' \mathbf{u}'} \rangle$	$\langle \overline{b'w'} \rangle$		$-\epsilon_{MKE}$
-7.3	0		-5.9

background shear or changes in stratification can make the waves unstable. In our simulations, a large inertial shear seems to trigger the observed wave breaking.

Upward-propagating wave packets, which are generally stable, based on the Richardson number criteria, break as they pass through a vertically sheared IO. The effect of IOs on the propagation of internal wave packets is described by Broutman and Young (1986), based on raytracing theory. Here, IOs play the role of a filter, separating long internal waves that manage to pass through IOs without any substantial change and short internal waves that are modulated by IOs, causing an increase in their vertical wavenumbers until they break. Almost all wave modes generated at the bottom have a scale shorter than the IO scale and are significantly affected by the IOs. Wave breaking is therefore confined to the depth range where there is significant IO shear.

The argument is a bit circular at this point, because we showed that the IOs are in turn driven by wave radiation and dissipation. The circularity is broken if we consider the initial value problem. Small IOs are generated during the initial transient (among other high-frequency transient waves). These IOs make the problem time dependent with the radiation of multichromatic wave packets described by the linear wave theory developed in section 3.



FIG. 8. Vertical profiles of the total energy dissipation rate (gray) and energy conversion from the mean subinertial flow into internal waves (black; W kg<sup>-1</sup>) for the  $\epsilon = 0.4$  simulation.



FIG. 9. Diagram of energy pathways. Energy conversion and dissipation values (mW m<sup>-2</sup>) are from the  $\epsilon = 0.4$  simulation. The values inside the boxes represent growth (time derivative) of kinetic energy.

The wave packets, predicted by (2), set the scale of the vertical momentum flux divergence and further reinforce the IOs within that depth range. The size of these packets depends on the vertical scales of internal wave modes and the number of modes involved: increasing both  $\beta$  and N/f (for example by decreasing f) allows for the radiation of a larger number of higher harmonics, which results in an increased characteristic wave packet scale, thicker IOs, and a thicker region characterized by wave breaking.

Finally, we diagnose the turbulent dissipation rate  $\varepsilon$  from the simulations. Figure 10 shows vertical profiles of time-averaged dissipation rates from different  $\epsilon$  simulations. Values of the dissipation rates integrated over 1-km depth above the bottom are shown in Fig. 11. The magnitude of the turbulent dissipation rate generally increases with  $\epsilon$ , as the internal wave amplitude grows and waves become more nonlinear and break. The dissipation rate is significantly intensified in the bottom 1 km, where most of the wave breaking takes place, and then decays in the ocean interior. The dissipation rate integrated over the bottom 1 km saturates at about 25 mW m<sup>-2</sup> for  $\epsilon \ge 0.7$ .

# 6. Conclusions

In this study, we used a weakly nonlinear theory and idealized numerical simulations to describe radiation and dissipation of internal waves generated by geostrophic flows over rough topography. A novel result of this paper is that there are two regimes of internal wave generation by geostrophic flows: quasi-stationary and time-dependent wave radiation regimes. In the quasi-stationary regime, internal waves radiate in the form of lee waves, with scales and amplitudes consistent with a linear theory for wave generation by steady flows. Lee waves radiate freely away from topography and result in low mixing rates above topography. In the time-dependent regime, we find that a resonant feedback between the large-scale flow and internal waves drives IOs within O(1) km above topography. A combination of the mean flow and IOs radiates time-dependent and multichromatic internal waves consistent with a linear theory for wave generation by oscillatory flows. In this regime, strong vertical shear associated with IOs results in enhanced wave breaking and high mixing rates above topography.

In the idealized problem considered here, we have chosen parameters that mimic geostrophic flows and topographic features found in the Drake Passage sector of the SO, one of the very few places where dissipation estimates from observations are available. We find that internal waves are radiated from a limited range of topographic horizontal scales, varying from about 600 m to 6 km, with most of the radiation resulting from topographic features of 1–3-km scales. The vertical scale of



FIG. 10. Vertical profiles of energy dissipation rate (W kg<sup>-1</sup>) from different simulations with different  $\epsilon$  values.



FIG. 11. Energy dissipation rate (mW m<sup>-2</sup>) integrated in the bottom 1 km.

the radiated internal waves is about 600 m, significantly shorter than the local ocean depth. The characteristic steepness parameter is 0.6—that is, topographic slopes are close to critical. At this steepness parameter, radiation of internal waves corresponds to the time-dependent and multichromatic wave radiation regime.

The associated wave dissipation rates are of O(10) mW m<sup>-2</sup>, in agreement with energy dissipation estimates inferred from observations by Naveira Garabato et al. (2004) and larger than those estimated by Kunze et al. (2006). Energy dissipation estimates from observations depend on the estimate of the strain/shear ratio which is, in the Drake Passage region, found to be larger than expected for internal waves. The large strain/shear ratio is interpreted by Kunze et al. (2006) as a high level of noise and was low-pass filtered resulting in smaller energy dissipation estimates. However, our study suggests that the large strain/shear ratio could result from the strong IO above the bottom in regions with steep topography, implying that the mixing rates could be as high as estimated by Naveira Garabato et al. (2004).

Our estimate of radiation by geostrophic motions in Drake Passage exceeds, by a factor of 10, the energy radiated by tides in the same region (based on data kindly provided by J. Nycander and described in the introduction). Equations (2) and (3) show that the energy radiation is proportional to the square of the bottom velocity, whether it is tidal or geostrophic. Differences in these two estimates reflect primarily that geostrophic flows are more energetic than tidal flows in large parts of the Southern Ocean—bottom geostrophic flows from LADCP data in Drake Passage and the Scotia Sea are O(10) cm s<sup>-1</sup> (Naveira Garabato et al. 2002, 2003), whereas barotropic tidal flows based on the Ocean Topography Experiment (TOPEX)/Poseidon Cross-Over Global Inverse Solution, version 3 (TPXO.3) model (Egbert et al. 1994) are smaller at O(2) cm s<sup>-1</sup> (Kunze et al. 2006). NF present a more thorough analysis of observations and confirm the estimates reported here.

The instability that triggers IOs at the bottom is essentially a parametric instability. Its characteristics are similar to the ones of the van der Pol oscillator, with the damping term represented by the divergence of the wave momentum fluxes. The superposition of a mean flow and an IO over the bottom topography generates a pair of waves: one is a lee wave, forced by the mean flow, and the other is a harmonic of the inertial frequency, forced by the IO. These two waves transfer energy to f through a triad interaction (McComas and Bretherton 1977); however, note that the triad would not be very efficient in the absence of the bottom boundary condition. The boundary conditions are key because, as these waves break and deposit their momentum, they reenergize the IO in the zonally averaged flow. A stronger inertial component of the zonally averaged flow energizes the initial pair of waves and the feedback continues. In a sense, this is a problem of wave-mean flow interaction rather than wave-wave interaction. As a result of the instability, the IO grows at the expense of the mean flow.

The vertical shear associated with the IOs promotes enhanced wave breaking by compressing the vertical scale of the waves radiated from the bottom topography. For a steepness parameter larger than approximately 0.4, this mechanism of the transferring of wave energy to small scales dominates over nonlinear wave–wave interactions (Polzin 2004). Nonlinear wave–wave interactions, instead, dominate at smaller values of the steepness parameter because IOs remain too weak to substantially affect the radiating waves.

The numerical simulation setup (2D and zonally periodic domain) is such that the IOs cannot radiate horizontally; therefore, they always stay in the region of strong geostrophic flow above rough topography, where they get reinforced by internal waves through the resonant feedback. In the ocean, however, IOs are modulated by the large-scale variations in the mean flow. This modulation imposes a horizontal scale on the oscillations, making them near-inertial, and allowing them to radiate both horizontally and vertically. Using a geostrophic eddy scale of O(100) km as the horizontal scale of near-inertial oscillations and 1 km as their vertical scale, we find, estimating wave group velocities, that it takes near-inertial waves roughly 20 days to radiate an eddy scale away. This time scale is an order of magnitude longer than the characteristic growth rate of O(2)days, associated with the resonant feedback mechanism. Over the 2-day time period, near-inertial oscillations radiate horizontally at a distance of only about 5 km. Furthermore, changes in the Coriolis frequency on this distance as a result of the beta effect and geostrophic flow shear are insignificant,  $\simeq 0.1\%$ . Based on these simple scaling arguments, we expect the feedback to occur also in 3D on a  $\beta$  plane. However, a more thorough analysis of the effect of a background vorticity gradient is left for a future study.

There is some observational support for enhanced IOs above rough topography in regions with high diapycnal mixing rates. Toole (2007) and Thurnherr et al. (2005) report a significant amount of shear variance derived from near-inertial motions, accompanied by strong subinertial currents on the western flank of the mid-Atlantic Ridge in the South Atlantic.

The results described in this study for the 2D problem have been extended to 3D in Nikurashin (2009). The extension of the theory is straightforward and there are no fundamental differences between 2D and 3D, except that the number of modes involved in the radiation problem increases substantially in 3D. Linear theory predicts accurately the energy radiation diagnosed from 3D numerical simulations with a sinusoidal bump. We have not extended the wave-mean flow feedback theory to 3D because of the algebraic complexity of the problem. However, numerical simulations show that the growth rate of IOs and the energy dissipation are similar to the 2D case.

Radiation and dissipation of internal waves generated by geostrophic flows over rough topography can be the primary mechanism responsible for enhanced abyssal mixing in the SO. The partial absence of meridional boundaries makes the SO a special place, where the geostrophic field has to equilibrate through dissipation at the bottom boundary. Quasigeostrophic turbulence theory suggests that this is achieved through an inverse cascade, where the geostrophic eddies develop a large barotropic component. In this study, we showed that bottom geostrophic flows can sustain enhanced diapycnal mixing O(1) km above topography, which is crucial for the dynamics of the lower cell of the meridional overturning circulation. These issues are further explored in a companion paper (NF), where the theory developed here is tested against observations.

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# APPENDIX A

#### **Scaling and Multiscale Expansion**

The expansion of equations governing the wave-mean flow interaction is best carried out in a nondimensional form. We nondimensionalize Eqs. (6)-(9) and the boundary conditions from (10) using the following scales:

$$\begin{split} t &\to f^{-1}t, \quad \mathbf{x} \to L_W \mathbf{x}, \quad z \to H_W z, \\ h_T &\to H_T h, \quad \mathbf{u} \to U_G \mathbf{u}, \\ w &\to U_G \frac{H_W}{L_W} w, \quad p \to f L_G U_G p, \\ b &\to \frac{f L_G U_G}{H_G} b, \quad \mathcal{D}_{m,b} \to f \mathcal{D}_{m,b}, \end{split}$$

where  $L_W$ ,  $H_W$ ,  $L_G$ , and  $H_G$  are the horizontal and vertical scales of the waves and geostrophic flow, respectively,  $H_T$  is the topographic amplitude, and  $U_G$  is the geostrophic velocity scale. With these scales, the governing equations take the following nondimensional form:

$$\begin{split} \mathbf{u}_t + \mathrm{Ro}\frac{L_G}{L_W}[(\mathbf{u}\cdot\mathbf{\nabla}_H)\mathbf{u} + w\mathbf{u}_z] + \hat{\mathbf{z}} \times \mathbf{u} &= -\frac{L_G}{L_W}\mathbf{\nabla}_H p + \mathcal{D}_m(\mathbf{u}), \\ \delta^2_W w_t + \delta^2_W \mathrm{Ro}\frac{L_G}{L_W}[(\mathbf{u}\cdot\mathbf{\nabla}_H)w + ww_z] &= -\frac{L_G}{L_W}p_z + \frac{L_G}{L_W}\frac{H_W}{H_G}b + \delta^2_W \mathcal{D}_m(w) \\ b_t + \mathrm{Ro}\frac{L_G}{L_W}[(\mathbf{u}\cdot\mathbf{\nabla}_H)b + wb_z] + Bu\frac{L_G}{L_W}\frac{H_W}{H_G}w = \mathcal{D}_b(b), \\ \mathbf{\nabla}_H \cdot \mathbf{u} + w_z &= 0. \end{split}$$

The lower- and upper-boundary conditions become

$$w|_{z=\epsilon h(\mathbf{x})} = \epsilon \mathbf{u} \cdot \nabla_H h(\mathbf{x}) \text{ and } w|_{z\to\infty} = 0.$$

The nondimensional numbers that appear in the problem are the Rossby number Ro =  $U_G/fL_G$  and the Burger number Bu =  $N^2 H_G^2/f^2 L_G^2$  of the geostrophic flow, the aspect ratio of the internal waves  $\delta_W = H_W/L_W$ , and the topographic steepness parameter  $\epsilon = H_T/H_W \approx NH_T/U_G$ .

In the ocean, geostrophic flows are characterized by a small Ro and vary on spatial scales much larger than internal waves, which allows us to impose the following scale separation:

$$\frac{L_W}{L_G} = \text{Ro}, \quad \frac{H_W}{H_G} = \text{Ro}$$

In addition, we assume that the aspect ratio of internal waves is small O(Ro)—that is, the waves are hydrostatic and the Burger number of the geostrophic flow is O(1). With this choice of parameters, the nondimensional equations and the boundary conditions can be rewritten as

$$\mathbf{u}_{t} + (\mathbf{u} \cdot \nabla_{H})\mathbf{u} + w\mathbf{u}_{z} + \hat{\mathbf{z}} \times \mathbf{u} = -\operatorname{Ro}^{-1}\nabla_{H}p + \mathcal{D}_{m}(\mathbf{u}),$$
(A1)

$$\operatorname{Ro}^{2}[w_{t} + (\mathbf{u} \cdot \nabla_{H})w + ww_{z}] = -\operatorname{Ro}^{-1}p_{z} + b + \operatorname{Ro}^{2}\mathcal{D}_{m}(w), \quad (A2)$$

$$b_t + (\mathbf{u} \cdot \nabla_H)b + wb_z + w = \mathcal{D}_b(b), \tag{A3}$$

$$\boldsymbol{\nabla}_{H} \cdot \mathbf{u} + \boldsymbol{w}_{z} = 0, \tag{A4}$$

$$w|_{z=\epsilon h(\mathbf{x})} = \epsilon \mathbf{u} \cdot \nabla_H h(\mathbf{x}), \quad w|_{z\to\infty} = 0.$$
 (A5)

We expand variables and scales in the small steepness parameter  $\epsilon$ , imposing that the leading-order flow  $\mathbf{u}^{(0)}$  is a superposition of a geostrophic flow  $\mathbf{u}^G$ , varying on slow temporal and spatial scales  $T_G$  and  $\mathbf{X}_G$ , and an inertial oscillation  $\mathbf{u}^I$  with an amplitude and phase varying on slow temporal and spatial scales  $T_I$  and  $\mathbf{X}_I$ , so that

$$\begin{split} \mathbf{u} &= \mathbf{u}^G(T_G, \mathbf{X}_G) + \mathbf{u}^I(t; T_I, \mathbf{X}_I) + \epsilon \mathbf{u}^{(1)} + \epsilon^2 \mathbf{u}^{(2)} \\ &+ \epsilon^3 \mathbf{u}^{(3)} + \cdots, \\ w &= w^G(T_G, \mathbf{X}_G) + w^I(t; T_I, \mathbf{X}_I) + \epsilon w^{(1)} + \epsilon^2 w^{(2)} \\ &+ \epsilon^3 w^{(3)} + \cdots, \\ p &= p^G(T_G, \mathbf{X}_G) + \epsilon^2 \mathrm{Rop}^I(t; T_I, \mathbf{X}_I) + \epsilon \mathrm{Rop}^{(1)} \\ &+ \epsilon^2 \mathrm{Rop}^{(2)} + \epsilon^3 \mathrm{Rop}^{(3)} + \cdots, \quad \text{and} \\ b &= b^G(T_G, \mathbf{X}_G) + \epsilon^3 b^I(t; T_I, \mathbf{X}_I) + \epsilon b^{(1)} + \epsilon^2 b^{(2)} \\ &+ \epsilon^3 b^{(3)} + \cdots, \end{split}$$

where  $\mathbf{u}^{(n)}$  are the higher-order motions depending on all scales of the problem. Consistent with the previous scaling, geostrophic flow  $\mathbf{u}^G$  is assumed to vary on the slow subinertial scales  $T_G = \operatorname{Rot}$  and  $\mathbf{X}_G = \operatorname{Rox}$ . In our problem, the inertial oscillation is forced by the vertical divergence of internal wave momentum fluxes, which are assumed to vary on  $O(\operatorname{Ro}^{-1})$  scales in the horizontal and  $O(\epsilon^{-1})$  scale in the vertical, implying that  $\mathbf{X}_I = \operatorname{Rox}$ ,  $Z_I = \epsilon z$ , and  $T_I = \epsilon^3 t$ . Assuming that Ro is  $O(\epsilon^4)$ , this choice of scales allows a clean separation between the geostrophic flow  $\mathbf{u}^G = \mathbf{u}^G[T^{(4)}, \mathbf{X}^{(4)}]$ , inertial oscillations  $\mathbf{u}^I = \mathbf{u}^I[t; T^{(3)}, X^{(4)}, Y^{(4)}, Z^{(1)}]$ , and internal waves  $\mathbf{u}^{(1)} =$  $\mathbf{u}^{(1)}(t, \mathbf{x}; \ldots)$ , where  $T^{(n)}$  and  $\mathbf{X}^{(n)}$  are the terms of an expansion of scales t and **x** in the small steepness parameter  $\epsilon$ .

The momentum and buoyancy dissipation operators are expended as

$$\mathcal{D}_{m,b} = \epsilon \mathcal{D}_{m,b}^{(1)} + \epsilon^2 \mathcal{D}_{m,b}^{(2)} + \epsilon^3 \mathcal{D}_{m,b}^{(3)} + \cdots$$

The dissipation operator  $\mathcal{D}_{m,b}^{(0)}$ , acting on the leadingorder flows  $\mathbf{u}^G$  and  $\mathbf{u}^I$ , which vary on the large spatial scales, is weak. For example, if we consider a Laplacian form of dissipation operator  $D_m^{(0)}(\mathbf{u}^{(0)}) = \nu \nabla^2 \mathbf{u}^{(0)}$  then the ratio of the dissipation and Coriolis terms in the governing equations scale as the Ekman number  $\mathrm{Ek} = \nu/(fH_G^2)$ . The Ekman number of geostrophic flows in the ocean is small  $O(\mathrm{Ro})$ . Hence, we assume that  $\mathcal{D}_{m,b}^{(0)}$  enters at higher orders in the expansion. Although the Laplacian form of the dissipation operators is more appropriate for this problem, it makes it difficult to solve the problem analytically. To make analytical progress, in section 3a we use dissipation operators in the form of a linear drag  $D_m^{(n)}(\mathbf{u}^{(n)}) = -\lambda \mathbf{u}^{(n)}$ , where  $\lambda$  is the dissipation rate. The dissipation rate of the leading-order flow is assumed to be weak,  $\lambda^{(0)} \ll f$ .

Expanding Eqs. (A1)–(A4) with the boundary conditions of (A5) and collecting  $O(\epsilon^0)$  terms, we obtain the following set of equations describing the evolution of the leading-order flow:

$$\mathbf{u}_{t}^{I} + \hat{\mathbf{z}} \times (\mathbf{u}^{G} + \mathbf{u}^{I}) = -\nabla_{\mathbf{X}^{(4)}} p^{G},$$
$$0 = -p_{Z^{(4)}}^{G} + b^{G}$$
$$w^{G} + w^{I} = 0.$$

Averaging these equations over the fast time scale t, we separate the balanced geostrophic flow  $\mathbf{u}_G$  from the evolution of the inertial oscillation  $\mathbf{u}_I$ . Corresponding sets of the equation in dimensional form are given in (12) and (13) for the geostrophic flow and inertial oscillation, respectively.

Collecting  $O(\epsilon^1)$  terms, we obtain a set of equations describing the evolution of internal waves generated by  $O(\epsilon^0)$  motions interacting with bottom topography. This set of equations is given in (14)–(18) in dimensional form. Averaging these equations over the small spatial scales we obtain the following equation for  $O(\epsilon^1)$  inertial oscillations:

 $\overline{\mathbf{u}_t^{(1)}} + \hat{\mathbf{z}} \times \overline{\mathbf{u}^{(1)}} = \mathcal{D}_m^{(1)}(\overline{\mathbf{u}^{(1)}}), \quad \overline{w^{(1)}} = 0,$ 

which is trivial because there is no forcing and  $\overline{\mathbf{u}^{(1)}} = 0$  at  $t = t_0$ .

To capture the evolution of  $O(\epsilon^0)$  inertial oscillation  $\mathbf{u}_I$ on the slow time scale  $T^{(3)}$ , we need to expand equations to higher orders. Collecting  $O(\epsilon^2)$  terms, we obtain

$$\begin{split} \mathbf{u}_{t}^{(2)} + \mathbf{u}^{(0)} \cdot \nabla_{\mathbf{x}} \mathbf{u}^{(2)} + \hat{\mathbf{z}} \, \times \, \mathbf{u}^{(2)} &= -\nabla_{\mathbf{x}} p^{(2)} + \mathcal{D}_{m}^{(2)} - w^{(1)} \mathbf{u}_{Z^{(1)}}^{I} - \mathbf{u}^{(1)} \nabla_{\mathbf{x}} \mathbf{u}^{(1)} - w^{(1)} \mathbf{u}_{z}^{(1)} \\ 0 &= -p_{z}^{(2)} + b^{(2)} - p_{Z^{(1)}}^{(1)}, \\ b_{t}^{(2)} + \mathbf{u}^{(0)} \cdot \nabla_{\mathbf{x}} b^{(2)} + w^{(2)} &= \mathcal{D}_{b}^{(2)} - \mathbf{u}^{(1)} \nabla_{\mathbf{x}} b^{(1)} - w^{(1)} b_{z}^{(1)}, \\ \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(2)} + w_{z}^{(2)} &= -w_{Z^{(1)}}^{(1)}. \end{split}$$

Averaging over small spatial scales, we obtain equations for the  $O(\epsilon^2)$  inertial oscillations:

$$\overline{\mathbf{u}_t^{(2)}} + \hat{\mathbf{z}} \times \overline{\mathbf{u}^{(2)}} = \mathcal{D}_m^{(2)}(\overline{\mathbf{u}^{(2)}}), \quad \overline{w^{(2)}} = 0.$$

Because there is no forcing and  $\overline{\mathbf{u}^{(2)}} = 0$  at  $t = t_0$ , this equation is trivial.

Finally, collecting  $O(\epsilon^3)$  terms, we get

$$\begin{split} \mathbf{u}_{t}^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_{\mathbf{x}} \mathbf{u}^{(3)} + \hat{\mathbf{z}} \times \mathbf{u}^{(3)} &= -\nabla_{\mathbf{x}} p^{(3)} + \mathcal{D}_{m}^{(3)} - \mathbf{u}_{T^{(3)}}^{I} - w^{(2)} \mathbf{u}_{Z^{(1)}}^{I} - -\mathbf{u}^{(1)} \nabla_{\mathbf{x}} \mathbf{u}^{(2)} - w^{(1)} \mathbf{u}_{z}^{(2)} - \mathbf{u}^{(2)} \nabla_{\mathbf{x}} \mathbf{u}^{(1)} \\ &- w^{(2)} \mathbf{u}_{z}^{(1)} - w^{(1)} \mathbf{u}_{Z^{(1)}}^{(1)}, \\ 0 &= -p_{z}^{(3)} + b^{(3)} - p_{Z^{(1)}}^{I} + b^{I} - p_{Z^{(1)}}^{(2)}, \\ b_{t}^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_{\mathbf{x}} b^{(3)} + w^{(3)} &= \mathcal{D}_{b}^{(3)} - b_{t}^{I} - \mathbf{u}^{(1)} \nabla_{\mathbf{x}} b^{(2)} - w^{(1)} b_{z}^{(2)} - \mathbf{u}^{(2)} \nabla_{\mathbf{x}} b^{(1)} - w^{(2)} b_{z}^{(1)} - w^{(1)} b_{Z^{(1)}}^{(1)}, \\ \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(3)} + w_{z}^{(3)} &= -w_{Z^{(1)}}^{(2)}. \end{split}$$

Averaging over the small spatial scales, we obtain the set of equations describing both the evolution of the  $O(\epsilon^3)$ inertial oscillations on the fast time scale and the evolution of  $O(\epsilon^0)$  inertial oscillations on the slow time scale. Equations in dimensional form are given in (19).

#### APPENDIX B

#### Wave Momentum Flux Divergence

A complete analytical expression for internal wave momentum flux divergence can be computed from the  $O(\epsilon^1)$  internal wave solution. If we write the solution for wave velocities as

$$w^{(1)} = -\sum_{n=-\infty}^{\infty} J_n(\beta) \operatorname{Im}(a_n e^{i\theta_n}), \quad a_n = h_T \sigma_n,$$
$$u^{(1)} = \sum_{n=-\infty}^{\infty} J_n(\beta) \operatorname{Im}(b_n e^{i\theta_n}), \quad b_n = h_T \sigma_n m_n k_T^{-1},$$
$$v^{(1)} = \sum_{n=-\infty}^{\infty} J_n(\beta) \operatorname{Re}(c_n e^{i\theta_n}), \quad c_n = h_T \sigma_n m_n k_T^{-1} f(\sigma_n - i\lambda)^{-1}$$

then using  $\mathcal{V} = u + iv$ , the zonally averaged wave momentum flux can be written as

$$\overline{\mathcal{V}^{(1)}w^{(1)}} = -\frac{1}{4} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_m(\beta) J_n(\beta) a_m \\ \times [(b_n^* + c_n^*)e^{i(\theta_m - \theta_n^*)} + (b_n - c_n)e^{-i(\theta_m^* - \theta_n)}].$$
(B1)

The phase of the wave momentum flux  $\theta_m - \theta_n^* = (m_m - m_n^*)z + (m - n)f(t - t_0)$  shows that the flux is time dependent and it oscillates at harmonics of the inertial frequency *f*. The corresponding vertical divergence of the wave momentum flux evaluated at z = 0 can be written as a superposition

$$\partial_z \overline{\mathcal{V}^{(1)}w^{(1)}} = \partial_z \overline{\mathcal{V}^{(1)}w^{(1)}}\Big|_{\omega=0} + \partial_z \overline{\mathcal{V}^{(1)}w^{(1)}}\Big|_{\omega=\pm f} + \cdots,$$

where the first-term steady component of the flux divergence is found by collecting terms with m = n in (B1), so that

$$\partial_z \overline{\mathcal{V}^{(1)} w^{(1)}} \Big|_{\omega=0} = \mathcal{A}(\beta)$$

$$= \sum_{n=-\infty}^{\infty} J_n^2(\beta) \operatorname{Im}(m_n) a_n [\operatorname{Re}(b_n) - i \operatorname{Im}(c_n)].$$

Collecting terms with  $m = n \pm 1$ , we obtain a timedependent component of the flux divergence oscillating at the inertial frequency  $\pm f$ ,

$$\partial_{z}\overline{\mathcal{V}^{(1)}w^{(1)}}\Big|_{\omega=\pm f} = -\mathcal{B}(\beta)e^{-if(t-t_{0})} - \mathcal{C}(\beta)e^{if(t-t_{0})}$$

where

$$\mathcal{B}(\beta) = \frac{i}{4} \sum_{n=-\infty}^{\infty} J_n(\beta) [J_{n-1}(\beta)a_{n-1}(b_n^* + c_n^*)(m_{n-1} - m_n^*) - J_{n+1}(\beta)a_{n+1}(b_n - c_n)(m_{n+1}^* - m_n)],$$
  
$$\mathcal{C}(\beta) = -\frac{i}{4} \sum_{n=-\infty}^{\infty} J_n(\beta) [J_{n-1}(\beta)a_{n-1}(b_n - c_n)(m_{n-1}^* - m_n) - J_{n+1}(\beta)a_{n+1}(b_n^* + c_n^*)(m_{n+1} - m_n^*)].$$

The amplitudes of the flux divergence  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  depend nonlinearly on  $\beta$ . In the limit of the small  $\beta$  this dependence can be simplified to

$$\begin{aligned} \mathcal{A}(\beta) &\approx A + O(\beta^2), \quad \mathcal{B}(\beta) \approx B\beta + O(\beta^2), \\ \mathcal{C}(\beta) &\approx C\beta + O(\beta^2), \end{aligned}$$

where A, B, and C are constants that depend on fixed external parameters of the problem. For example, the steady component of the wave momentum flux divergence becomes

$$\begin{split} A &= h_T^2 U_G^2 k_T \operatorname{Re}(m_0) \operatorname{Im}(m_0) \\ & \times \Biggl\{ 1 - i \Biggl( \frac{f U_G k_T}{U_G^2 k_T^2 + \lambda^2} \Biggr) \Biggl[ \frac{\lambda}{U_G k_T} + \frac{\operatorname{Im}(m_0)}{\operatorname{Re}(m_0)} \Biggr] \Biggr\}, \end{split}$$

which is the momentum flux divergence of the steady, damped lee waves with rotation. This flux divergence is zero in the limit of no damping. Expressions for *B* and *C* are somewhat more convoluted and not shown here, but they can be easily obtained from  $\mathcal{B}(\beta)$  and  $\mathcal{C}(\beta)$  using

$$J_0(\beta) \approx 1 + O(\beta^2), \quad J_{\pm 1}(\beta) \approx \pm \frac{1}{2}\beta + O(\beta^2).$$

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