Statistics of dispersion in flows with coherent structures

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Abstract. Float trajectories in the ocean are shown to have moments of displacements with large-t behavior $\langle |x|^p \rangle \sim t^{\gamma_p}$ for times below an year. The study of γ_p as a function of p provides a more complete characterization of ocean dispersion than does the single number γ_2. Also at long times, the core of the displacement distribution relaxes to a self-similar profile, while the tails consisting of floats which have experienced exceptional displacements, are not self-similar. Depending on the region considered, the effect of the tails can be negligible, and then γ_p is a linear function of p (strong self-similarity). But if the tails are important then γ_p is a non-trivial function of p (weak self-similarity). In the weakly self-similar case, the low moments are determined by the self-similar core, while the high moments are determined by the non-self-similar tails. The popular exponent γ_2 may be determined by either the core or the tails. Implications of these results for parameterization of ocean dispersion in numerical models are discussed.

Introduction

An outstanding problem in large-scale ocean dynamics is the understanding, characterization, and representation of tracer transport by coherent mesoscale structures, such as jets, intense vortices, eddies, planetary waves. In coarse resolution models used for climate studies the mesoscale transport is parameterized as an enhanced eddy diffusivity D. This approach is formally valid only for times longer than the decorrelation time of the coherent mesoscale structures of order of a few months up to a few years [Taylor, 1921; Freeland et al., 1975; Veneziani et al., 2004]. The goal of this paper is to discuss whether alternative mathematical models can be developed to parameterize mesoscale transport at shorter times.

Taylor [1921] provided the first formal analysis of dispersion of a cloud of tracer particles in a turbulent velocity field. In particular he showed that at times longer than the decorrelation time of the Lagrangian velocity, the mean square displacement of a cloud of particles, or equivalently the area of a tracer patch[1], grows linearly with time and the rate of increase is given by the eddy diffusivity. In the eighty years since Taylor's pioneering work, it has been noted that in most turbulent flows normal diffusion is achieved only after extremely long transients, because long-lived coherent structures spontaneously emerge and generate long tails in the covariance function of the Lagrangian velocity. During these transients dispersion is *anomalous*, i.e. the mean square particle displacement does not grow linearly in time. There is mounting theoretical [Young, 1988; Ferrari et al., 2001; Zaslavsky, 2002; Reynolds, 2002a] and observational [Solomon et al., 1994; Weeks et al., 1996; Cardoso and Tabeling, 1996] evidence that generalizations of Taylor's approach can be derived to study anomalous dispersion.

Anomalous dispersion has been observed in the ocean. Rupolo et al. [1996] found evidence of anomalous dispersion in float trajectories from the Western North Atlantic for times between a few days and three months. Berloff and McWilliams [2002] and Reynolds [2002b] studied particle and tracer dispersion in high resolution numerical simulations of wind-driven ocean gyres. They found that in many regions dispersion was anomalous for up to an year. Berloff and McWilliams [2002] went on to propose mathematical models to parameterize this anomalous dispersion. A major limitation of these studies is that they focused exclusively on second order statistics like the mean square particle displacement and the covariance function of the Lagrangian velocity. In this paper we show that anomalous dispersion cannot be fully characterized by second order statistics. A consideration of *lower and higher* order statistics is necessary to properly quantify anomalous dispersion and derive accurate parameterizations.

The paper is structured as follows. First we introduce the dichotomy between normal and anomalous diffusion in terms of second order statistics of dispersion. Then we show that a consideration of lower and higher order statistics provides a better characterization of dispersion. Finally these concepts are illustrated with a stochastic model, the "non-Markovian telegraph

[1] The equivalence between the statistics of particle displacements and tracer concentration holds as long as the tracer concentration associated with tagged water parcels does not change much as a result of molecular mixing over the time for which the Lagrangian correlation time is significant.

model", and with an analysis of float trajectories from the Western North Atlantic.

Normal versus anomalous diffusion

Let us consider the Lagrangian time series, such as the x-velocity, of a tagged fluid particle, $u(t)$, as a function of time. We limit the discussion to one spatial dimension for simplicity, but the results apply to any number of dimensions. The distinction between normal and anomalous diffusion can be understood by examining the rate at which the velocity covariance decreases to zero. Normal diffusion occurs if the velocity covariance decreases rapidly, while anomalous diffusion results from processes in which particles move coherently for long times with infrequent changes of direction. This distinction is quantified by the tail behaviour of the velocity autocovariance function. For example, if the covariance function decays exponentially then there is normal diffusion, whereas if the covariance function decays algebraically then there is the possibility of anomalous diffusion.

The definition of anomalous diffusion is based only on the behaviour of the mean square displacement of particles about the center of mass, $\langle x^2 \rangle$, where angular brackets $\langle\ \rangle$ denote an ensemble average over many realizations of the dispersion process. But we usually want to know more about the distribution of particles than simply the second moment. In the case of normal diffusion, detailed information concerning the distribution is obtained by solving the diffusion equation,

$$c_t = D c_{xx} \tag{1}$$

The goal of this paper is to discuss what statistical information is required to develop continuum models, analogous to (1), which provide the same detailed information for anomalously diffusing particles.

Normal diffusion

We have indicated above that the crudest measure of dispersion of a cloud of tagged fluid particle is the mean square displacement about the center of mass, $\langle x^2 \rangle$. We can calculate the rate of change of $\langle x^2 \rangle$ by first noting that,

$$\frac{\mathrm{d}x^2}{\mathrm{d}t} = 2\int_0^t u(t)u(t')\,\mathrm{d}t'. \tag{2}$$

We now ensemble average the right part of equation (2). If we assume that the Lagrangian velocity is statistically stationary, $\langle u(t)u(t')\rangle$ depends only on the time difference $t-t'$. Thus, we introduce the covariance function,

$$\mathcal{C}(t-t') \equiv \langle u(t)u(t')\rangle, \tag{3}$$

and, after a change of variables, write the ensemble average of (2) as

$$\frac{\mathrm{d}\langle x^2\rangle}{\mathrm{d}t} = 2\int_0^t \mathcal{C}(t')\,\mathrm{d}t'. \tag{4}$$

Equation (4) is Taylor's formula, which relates the variance in particle displacement $\langle x^2 \rangle$ to an integral of the Lagrangian velocity covariance function $\mathcal{C}(t)$. Taylor's formula is one of the most important results on dispersion. However it is rarely pointed out that it cannot be easily extended to moments other than the second. Little progress has been made in relating $\langle |x| \rangle$, or other moments, such as $\langle x^4 \rangle$, to velocity statistics. This will turn out to be a key limitation when applying Taylor's approach to study anomalous diffusion.

Taylor envisioned situations in which the covariance function $C(t)$ decreases rapidly to zero as $t \to \infty$, so that the integral in (4) converges. In this case, the dispersion of the ensemble of particles at large times is characterized by a diffusivity D given by,

$$D = \int_0^\infty \mathcal{C}(t')\,\mathrm{d}t'. \tag{5}$$

Integrating both sides of (4) over the time we have

$$\langle x^2 \rangle = 2\int_0^t (t-t')\mathcal{C}(t')\,\mathrm{d}t'. \tag{6}$$

The results above relate the variance in particle displacement $\langle x^2 \rangle$ to an integral of the Lagrangian velocity autocovariance function $\mathcal{C}(t)$. If $\mathcal{C}(t)$ decreases to zero as $t \to \infty$, the dispersion of the ensemble at large times is characterized by a diffusive growth rate $\langle x^2 \rangle \sim 2Dt$.

Anomalous diffusion

Dispersion experiments over the last twenty years have revealed behaviors which are much richer than those suggested by the arguments of Taylor. There are numerous examples of processes for which the growth of variance is described by a power law

$$\langle x^2 \rangle \propto t^\gamma. \tag{7}$$

Sometimes $\gamma = 1$ (normal diffusion), while in other cases $\gamma \neq 1$. If $\gamma \neq 1$ the process is referred to as anomalous diffusion.

Typically one thinks of processes that have a $C(t)$ which decreases to zero as $t \to \infty$, so that the integrals in (4) and (6) converge to nonzero values. Anomalous diffusion corresponds to situations in which $\mathcal{C}(t)$ decreases so slowly that the integral in (4) diverges. This can happen when the covariance function $\mathcal{C}(t)$ has a *long tail*, that is $\mathcal{C}(t) \sim c\ t^{1-\nu}$ with $1 < \nu < 3$ as $t \to \infty$.

When diffusion is anomalous the diffusivity D doesn't exist anymore. However, it still follows, integrating (6) with $\mathcal{C}(t) \sim c\ t^{1-\nu}$, that

$$\langle x^2 \rangle \sim \frac{c\ t^{3-\nu}}{(3-\nu)(2-\nu)}. \tag{8}$$

The case in which $\langle x^2 \rangle = 2Dt$, that is with $\nu = 2$, is considered *normal* diffusion. Thus it seems appropriate to define,

- *superdiffusion* when $1 < \nu < 2$; the RMS displacement of the ensemble grows faster then linearly with time, as $3 - \nu$ is greater than one.
- *subdiffusion* when $2 < \nu < 3$; the condition that $2 < \nu$ ensures that the integral of $C(t)$ converges to zero; the second inequality, $\nu < 3$, ensures that the integral in (6) diverges. The exponent $3 - \nu$ is less than one.

At first glance, the two possibilities of *subdiffusion* and *superdiffusion* appear as unlikely exceptions to the more natural cases where D is given by a convergent integral. However there are numerous examples in fluid mechanics in which both super- and subdiffusion are observed experimentally or computationally. Long tails in $C(t)$ are generally the result of long-lived coherent structures and cannot be dismissed as unlikely pathologies.

Strongly versus weakly self-similar diffusion

The dichotomy between normal and anomalous diffusion is based solely on the behavior of the second moment. The overwhelming majority of studies on anomalous diffusion are concerned mostly with that single descriptor of dispersion. However, one often wants to know more than the evolution of $\langle x^2 \rangle$. In general, one would like to describe the evolution of the overall distribution of particles. The problem can be tackled by considering the distribution of particles released at a point. In mathematical jargon this corresponds to finding the Green's function or propagator of the dispersion process. The propagator can then be used to obtain solutions for arbitrary initial particle distributions. It is usually not possible to obtain the propagator exactly, though asymptotic methods often provide useful approximations in the form of similarity solutions. That is to say, as $t \to \infty$, the propagator collapses to the self-similar form

$$c(x,t) \approx t^{-1/\nu} \mathcal{C}(x/t^{1/\nu}) \,. \tag{9}$$

In the case of normal diffusion $\gamma_2 = 1$, $\nu = 2$ and $\mathcal{C}$ is a Gaussian. In the superdiffusive case there are examples in which $\mathcal{C}$ is a Lévy density [e.g., see Zumofen and Klafter, 1993]. In the subdiffusive case, $\mathcal{C}$ is neither Lévy nor Gaussian [e.g., see Young, 1988].

The approximation in (9) is valid only in a *central scaling region* (CSR). The nonscaling tails of the distribution are occupied by particles which have experienced exceptionally large displacements. Sometimes tails are important even though they contain few particles: the second moment $\langle x^2 \rangle$ might be dominated by a few large terms corresponding to tail particles. This problem is strikingly demonstrated when $\mathcal{C}(\xi)$ is a Lévy density which has, for $\xi \gg 1$, a slow algebraic decay: $\mathcal{C}(\xi) \sim \xi^{-\nu}$ with $1 < \nu < 2$. Thus, in the Lévy case, the second moment of the similarity approximation (9) diverges. But in a simulation or experiment $\langle x^2 \rangle$ is always finite and this problem with the Lévy density is simply the result of incorrectly applying (9) outside of the CSR.

To summarize, the tails are not described by (9), and there are several important questions suggested by this observation. For instance, we have introduced two exponents, γ_2 and $1/\nu$. How are they related? The naive answer is that $\gamma_2 = 2/\nu$. In fact, $\gamma_2 = 2/\nu$ is correct if $\mathcal{C}$ is a Gaussian, and also in the subdiffusive example in Young [1988]. But $\gamma_2 \neq 2/\nu$ if $\mathcal{C}$ is a Lévy density. How representative of the dispersion of a typical particle is the single statistic $\langle x^2 \rangle$, and the exponent γ_2? Given the results of an experiment or simulation, how can we detect the existence of a CSR and determine the two exponents γ_2 and $1/\nu$? Is the similarity solution in (9) the solution of a partial differential equation, or perhaps a "fractional kinetic equation" such as those derived in Zaslavsky [2002]?

Moments and strong versus weak self-similarity

The general moments, $\langle |x|^p \rangle$, provide information about both the CSR and the non-scaling tails. Small values of p sample the CSR, while larger values of p sample the tails. Given the great interest in turbulent structure functions of fractional order [e.g. Frisch, 1995], it is surprising that $\langle |x|^p \rangle$ has attracted only sporadic attention as a descriptor of dispersion. We located very few papers which discuss $\langle |x|^p \rangle$ in the context of anomalous diffusion [e.g. Aranson et al., 1990; Castiglione et al., 1999; Andersen et al., 2004] and none in the context of geophysical flows.

Using the behavior of $\langle |x|^p \rangle$ as $t \to \infty$ Ferrari et al. [2001] propose to make a distinction between *strong self–similarity* and *weak self–similarity.* Given a power-law growth of the p'th moment,

$$\langle |x|^p \rangle \sim t^{\gamma_p} \,, \tag{10}$$

they say that a process is strongly self–similar if *all* mo-

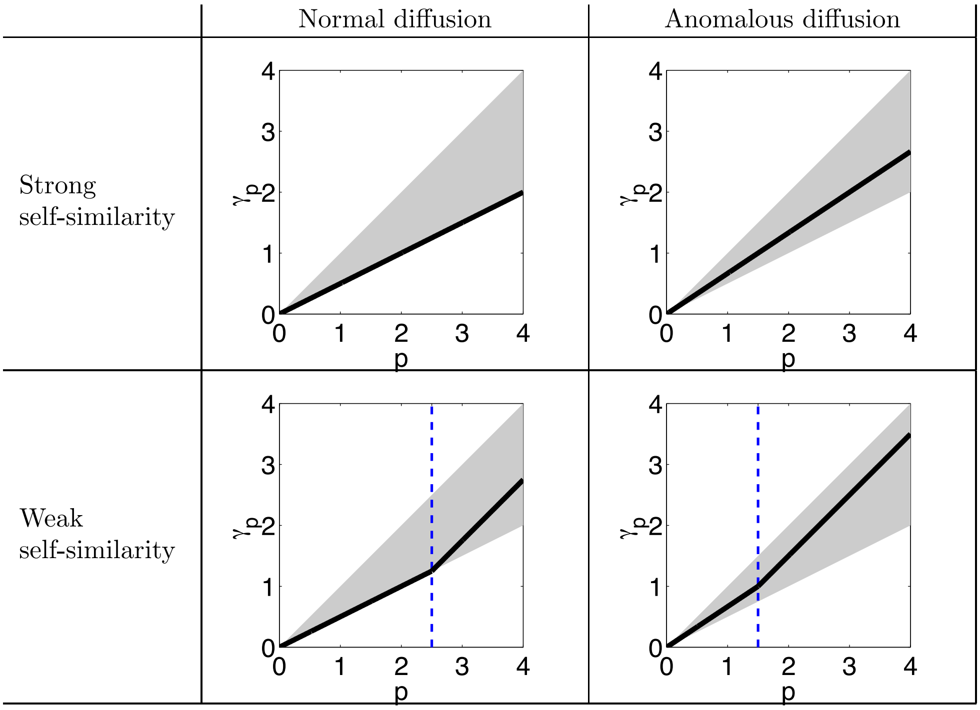

Figure 1. Classification of processes as normally versus anomalously diffusive, and weakly versus strongly self-similar. In the four panels the shaded wedge indicates the region between normal diffusion, $\gamma_p = p/2$, and ballistic separation $\gamma_p = p$. In the bottom row the break in slope is at $p = \nu$. The most elementary case is the top left panel, which is exemplified by the Markovian telegraph process. The most complicated case is the bottom right panel which exemplified by the non-Markovian telegraph process.

ments satisfy the scaling law suggested by (9). In other words, if $\gamma_p = p/\nu$ for all p then the process is strongly self–similar. If γ_p is a more interesting function of p, such as the example in the bottom left panel of Fig. 1d, then the process is weakly self-similar. While all diffusive processes have nonscaling tails which defeat (9), this defeat is particularly sharp if the process is weakly self-similar. The function γ_p, as opposed to the single number γ_2, provides more information and the possibility of more stringent comparison between theory, simulation and observation. Furthermore a characterization of the CSR is essential to derive mathematical models of the dispersion process and this cannot be done in terms of γ_2 only.

The dichotomy between strong and weak self-similarity is independent of the dichotomy between normal and anomalous diffusion. Thus, as indicated in Fig. 1, there are four cases which might occur. In this article we present several models which, depending on parameter settings, fall into each of the four boxes in Fig. 1.

The telegraph model: a theoretical case study

The telegraph model is a simple example of a random walk continuous in time. Here we use it as a tutorial example to study the properties of dispersion processes. The model is particularly instructive because one can compute the scaling for all moments and obtain the CSR exactly. We introduce both a Markovian and a non-Markovian telegraph model to represent respectively flows without and with long-lived coherent structures. Then we use the models to illustrate the dichotomy between normal and anomalous diffusion versus the dichotomy between strongly and weakly self-similar diffusion.

Markovian telegraph model

In a telegraph process, the velocity of a particle, $u(t)$, can have only one of two possible values, $+U$ and $-U$. The velocity of each particle flips randomly back and forth between $\pm U$ with a transition probability $\alpha/2$ per unit time. This means that in a time dt a fraction αd$t/2$ of the particles switches velocity. If the transition rate $\alpha/2$ is constant we can say that a particle has no memory of when it first arrived in its present state.

Thus this telegraph model is Markovian.

The velocity covariance function and diffusivity for the Markovian telegraph model are [e.g. Young, 1999],

$$C(t) = U^2 e^{-\alpha t}, \qquad D = \frac{U^2}{\alpha}. \tag{11}$$

The exponentially decaying covariance function ensures that D is finite and that the displacement variance ultimately grows diffusively, i.e. $\langle x^2 \rangle \sim 2Dt$ for $t \gg 2/\alpha$. For the Markovian telegraph model we can explicitly compute the evolution equation for the total concentration of particles $c(x,t)$,

$$c_{tt} + \alpha c_t - U^2 c_{xx} = 0. \tag{12}$$

On large, slowly evolving length-scales scales one can neglect the term c_{tt} in (12) and so obtain the diffusion equation, with the diffusivity U^2/α, as an approximation of (12).

Using the method of Morse and Feshbach [1953], the solution of (12) with the initial conditions $c(0,x) = \delta(x)$ and $c_t(0,x) = 0$ is

$$c(x,t) = \frac{1}{2U}\left(\frac{\partial}{\partial t} + \alpha\right) e^{-\alpha t/2} I_0 \left[\frac{\alpha}{2U}\sqrt{U^2t^2 - x^2}\right], \tag{13}$$

for $|x| \le Ut$ and I_0 is a modified Bessel function. $c(x,t) = 0$ for $|x| > Ut$. The Gaussian similarity approximation, which applies in the CSR, is obtained by taking the double limit $t \to \infty$ with $|x|\alpha^{1/4}/Ut^{3/4} \to 0$ in (13). Thus, the tails of $c(x,t)$ span the region

$$(U/\alpha)\,(\alpha t)^{3/4} < |x| < U\,t\,. \tag{14}$$

The Green's function in (13) is plotted using the similarity variable $x/\sqrt{t}$ in Fig. 2. The nonscaling tails fall below the Gaussian approximation except for δ-function peaks at $x = \pm Ut$. These ballistic peaks consist of particles which have traveled with constant velocity since $t = 0$. However, the complete tail-structure of C is more complicated than simply a pair of ballistic peaks at $x = \pm Ut$: the Gaussian similarity approximation fails, and the tails begin, at $x \sim t^{3/4} \ll t$. But, despite the asymptotically expanding zone in (14), the moments have $\langle |x|^p \rangle \sim t^{p/2}$ for all p. Thus, this elementary telegraph model is both normally diffusive and strongly self-similar. The tail structure is not self-similar, but the failure of similarity is so mild that as $t \to \infty$ all moments are determined by the CSR. This is the scaling shown in the upper left panel of Fig. 1.

Non-Markovian telegraph model

Ferrari et al. [2001] developed a generalization of the telegraph model which can be used to illustrate all four case in Fig. 1. In this generalization particles switch randomly between moving with $u(t) = +U$ and $u(t) = -U$. The transition probabilities between these states is function of the time since the last transition. In other words, each particle carries an "age", a, which is the time elapsed since the particle transitioned into its present state. The introduction of memory of the past makes the model non-Markovian and admits the possibility of anomalous and weakly self-similar diffusion.

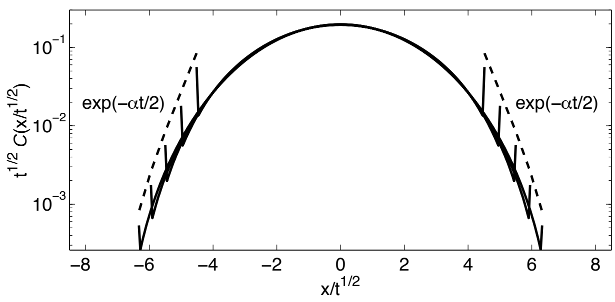

Figure 2. The Green's function in (13) at five evenly spaced times with $10 < \alpha t < 20$. In this semi-log plot the Gaussian similarity approximation is a parabola and as t increases the curves collapse onto this parabola. The strongest symptom of the tails are δ-function peaks at $x = \pm Ut$. These ballistic peaks are produced by particles which have traveled with constant velocity since $t = 0$. The number of these particles decays like $\exp(-\alpha t/2)$, as indicated by the dashed lines.

The formulation of the non-Markovian telegraph model boils down to specifying the dependence of the transition rate α on the age a. As far as scaling exponents of moments and CSR are concerned, only the $a \gg 1$ structure of α matters. The asymptotic form of α is determined with a simple dimensional argument. The transition probability α has the dimensions of inverse time. If the only time-scale relevant for long-lived particles is the particle age, a, then it follows that α is inversely proportional to a, i.e., $\alpha \sim \nu/a$. There is observational support for this scaling law. For example Solomon et al. [1994] estimated the transition probability of the direction of propagation of particles undergoing anomalous diffusion in dispersion experiments in a rotating annular tank. They found that α depended inversely with the time since the last transition. Ferrari et al. [2001] further showed that the non-Markovian telegraph model with $\alpha \sim \nu/a$ did reproduce well the experimental results of Solomon and collaborators.

With the choice $\alpha \sim \nu/a$, the velocity covariance for the non-Markovian telegraph model at long times scales as, $C(t) \sim c\, t^{1-\nu}$ [Ferrari et al., 2001]. For $1 < \nu < 2$ the integral of $C(t)$ diverges and there is anomalous diffusion,

$$\langle x^2 \rangle \sim \frac{ct^{3-\nu}}{(3-\nu)(2-\nu)}. \qquad (15)$$

If $\nu > 2$ then there is normal diffusion,

$$\langle x^2 \rangle \sim 2Dt + O(t^{3-\nu}). \qquad (16)$$

Subdiffusion appears in the telegraph model if particles, in addition to alternating between flights with velocities $u(t) = \pm U$, are also allowed to stick and spend time without moving, $u(t) = 0$ [Ferrari et al., 2001]. In a geophysical context the flights could represent advection by jets and the pauses trapping by stationary vortices. We do not pursue these embellishments because they are not essential for illustrating the fundamental properties of the non-Markovian telegraph model.

Ferrari et al. [2001] derive analytically the scaling for all moments $\langle |x|^p \rangle$ for the non-Markovian telegraph model. They find that γ_p is a piecewise–linear function of p and the break in slope occurs at $p = \nu$ as shown in Fig. 3. The break in slope at $p = \nu$ is produced by an exchange of dominance between the majority of particles in the CSR, which determine the moments with $p < \nu$, and the exceptional particles in the tail which determine the moments with $p > \nu$. The tail particles have experienced *almost* ballistic motion, i.e. unidirectional propagation at constant speed, which is why the line $\gamma_p = p + \nu - 1$ is parallel, but below, the pure ballistic law $\gamma_p = p$.

Solutions for the propagator are difficult to obtain analytically and the interested reader is referred to Ferrari et al. [2001]. In Fig. 4 we show the concentration $c(x,t)$ at various times from a numerical integration of the non-Markovian telegraph model with initial conditions $c(0,x) = \delta(x)$ and $c_t(0,x) = 0$. The solution is plotted using the similarity scaling in (9). The scaling collapses the concentration in a CSR where the distribution is well approximated by a Lévy density. The CSR determines the evolution of moments $p < \nu$. The disagreement between the Lévy density and the numerical simulation in the tails is because the Lévy approximation is not valid at large x. The disagreement is particularly striking at the δ-peaks at $x = \pm Ut$ which correspond to particles that have never transitioned from their original state. These peaks decay algebraically in time as $t^{1-\nu}$ and determine the scaling of moments $p > \nu$.

We have shown that in the non-Markovian telegraph model γ_p is a piecewise–linear function of p. We are not claiming that all weakly self–similar processes have the piecewise–linear relation in Fig. 3. But piecewise–linear γ_p is probably the simplest form of weak self-similarity:

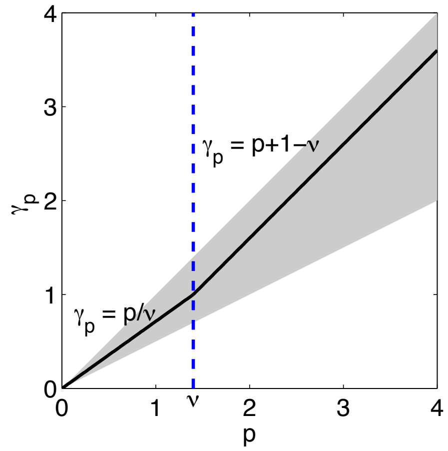

Figure 3. A schematic illustration of a non-Markovian telegraph model which is both weakly self-similar and superdiffusive. The exponent γ_p is a piecewise linear function of p with a break at $p = \nu$. The shaded wedge is the region between the diffusive law $\gamma_p = p/2$ and the ballistic law $\gamma_p = p$. Low-order moments, $p < \nu$, are determined by a central scaling region, in which (9) applies. The higher moments, $p > \nu$, are determined by the nonscaling tails of the concentration profile. Since $\nu < 2$ the second moment is determined by the tails of the concentration and determination of γ_2 does not provide information about the dispersion of a typical particle in the central scaling region.

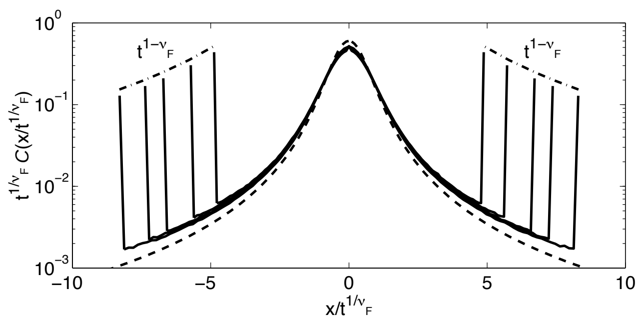

Figure 4. Particle concentration $c(x,t)$ at a few selected times produced by a non-Markovian telegraph model. The amplitude of the concentration and the x-axis are rescaled with the factor t^{1/ν_F} to show the self-similar nature of the CSR. The dashed line is the self-similar Lévy density to which the concentrations should converge at large times. The dash-dotted lines show the algebraic decay of the δ-peaks at $x = \pm Ut$.

the break in slope is a clean signature of the exchange of dominance between the CSR and the tails of the distribution of particles. The study of a set of moments is therefore a useful tool to identify whether there is a CSR that satisfies the similarity scaling in (9). This information can then be used to derive equations that reproduce the scaling of the diffusive process. Zaslavsky [2002] shows that the long-time behavior of the propagator core of diffusive processes that satisfy (9) can be described by Fractional-Fokker-Planck-Kolmogorov (FFPK) equations which are integro-differential equation with algebraically decaying kernels. FFPK are the natural extension of the diffusion equation for non-normally diffusive processes. Much alike the diffusion equation, they are accurate in a CSR but not in the tails of the distribution.

Dispersion of floats in the Western North Atlantic: an oceanographic case study

There have been a few studies documenting anomalous diffusion in the ocean [Rupolo et al., 1996; Berloff and McWilliams, 2002; Reynolds, 2002b]. These authors find that diffusion can be anomalous in the ocean for up to an year as a result of mesoscale coherent structures that introduce long-memory effects in the dispersion of tracers and floats. These studies focused on second order moments. We are not aware of any study that analyzed other moments to determine whether diffusion in strongly or weakly self-similar during these transients and whether there is CSR that can be used to derive a parameterization of the diffusive process. Here we attempt such a calculation using float data from the Western North Atlantic.

The dataset analyzed in this section is archived at the Subsurface Float Data Assembly Center (WFDAC) at Woods Hole. We use 105 float trajectories from the NAC and ACCE experiments [**?**]etails of the experiments can be found in[]LaCasce00. These were obtained with neutrally buoyant subsurface drifters, ballasted for a few density surfaces within the main thermocline (100-900 m) and tracked acoustically using the deep sound channel. Position fixes were made 1-3 times a day. All floats lasted at least 250 days. The NAC floats are in the Newfoundland Basin, that is between the Great Banks and the mid-Atlantic ridge. The ACCE region is in the central North Atlantic, straddling the mid-Atlantic ridge. The North Atlantic Current and its associated mesoscale eddies dominate the flow [Zhang et al., 2001], and the region is richly energetic. LaCasce and Bower [2000] show that the eddy statistics in these regions are fairly uniform and the ensemble of trajectories can be considered statistically homogeneous. The full set of float trajectories is shown in Fig. 5.

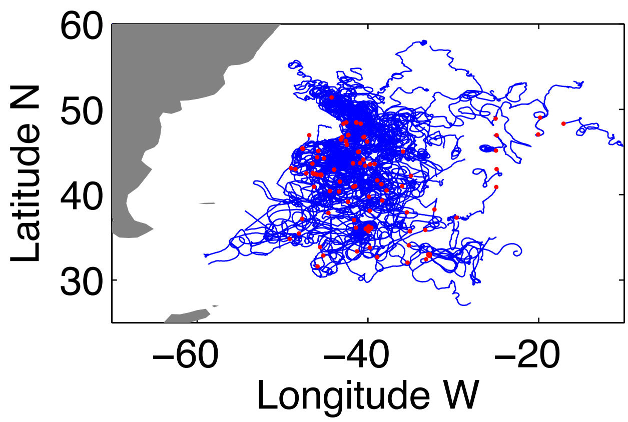

Figure 5. Float trajectories used in this analysis. The dataset were collected as part of the NAC and ACCE experiments. The red dots indicate where the floats were released.

Our goal is to quantify the float dispersion by transient mesoscale eddies. Hence the first task is to remove from the full velocity the the mean climatological flow $\bar{u}$ to obtain the eddy velocity $u' \equiv u - \bar{u}$. The eddy velocity so defined will include different kinds of fluctuation phenomena, like vortices due to flow instabilities, wave fluctuations, and possible seasonal-to-interannual variability. Following Davis [1991] and given that the northwest Atlantic can be characterized by regions with quasi-homogeneous statistical properties, we estimated the mean flow by binning the float velocities in boxes of $1° \times 1°$ and then averaging all measurements within each box. The size of the binning boxes was chosen as the most sensible tradeoff between the importance of resolving both spatial shears of the mean flow and eddy scales on the order of the internal Rossby radius of deformation, and the necessity of keeping a high enough data density per bin to guarantee statistical significance of the results. Veneziani et al. [2004] found that this approach compares favorably versus alternative techniques and provides a robust estimate of the mean flow in the region here considered.

The eddy float displacements are then computed integrating the equations,

$$\frac{dx}{dt} = u(t) - \bar{u}(x, y), \qquad \frac{dy}{dt} = v(t) - \bar{v}(x, y). \qquad (17)$$

The moments are computed as,

$$\langle |x|^p \rangle(t) = \frac{1}{N} \sum |x_n|^p,$$
$$\langle |y|^p \rangle(t) = \frac{1}{N} \sum |y_n|^p,$$

where the sum is over the 105 float trajectories. The moment exponents, γ_p, are then estimated by linear least squares fit between $\log\langle|x|^p\rangle$ and $\log t$ (and equivalently for meridional displacements). We find that there is a robust scaling for times between 50 and 250 days. For shorter times the moments do not show a power law scaling. The scaling for moments with $0 < p < 4$ are shown in Fig. 6. Zonal displacements follow the scaling for strongly self-similar normal diffusion with $\gamma_p = p/2$. Meridional displacements are instead weakly self-similar and subdiffusive. These results are consistent with the studies of Rupolo et al. [1996], based on analysis of float data, and of Berloff and McWilliams [2002], based on numerical simulations of a wind-driven gyre. Both found normal diffusion in the zonal direction and subdiffusive behavior in the meridional direction for up to an year in the northern part of the subtropical gyre. They interpreted the subdiffusive behavior as a signature of the barrier to transport due to meridional PV gradients. However both studies considered exclusively the growth rate of the second moment. Fig. 6 shows that there is a CSR for $0 < p < 1$ where $\gamma_p = p/2$. This suggests that the CSR scales like a Gaussian and is well described by the regular diffusion equation. The subdiffusive behavior is not associate with a CSR but with the scaling of the tails of the particle distributions.

The results presented in this section are preliminary. It is possible that the weak self-similarity in the scaling of the meridional moments can be partly attributed to lack of convergence of the statistics over the time window considered. However similar issues can be raised for the results referenced above. More importantly is that by focusing on a set of moments, we were able to identify a CSR which displays the diffusive law $\gamma_p = p/2$ both in the zonal and in the meridional directions. It is the tails of the particle distributions that display anomalous behavior in the region considered.

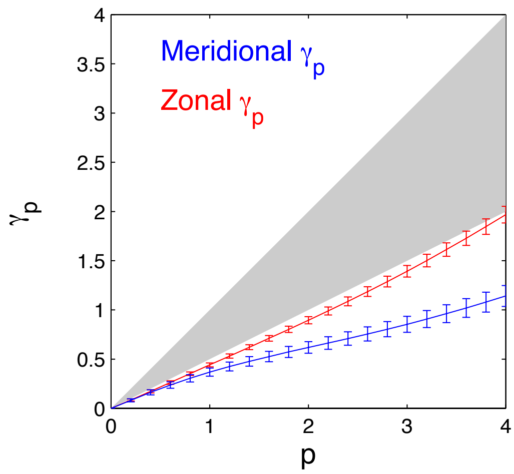

Figure 6. Scaling for the zonal and meridional eddy displacements for the float ensemble shown in Fig. 5. The eddy displacements are computed according the the evolution equations in (17). The scaling exponent γ_p is shown for moments $0 < p < 4$. The shaded wedge is the region between the diffusive law $\gamma_p = p/2$ and the ballistic law $\gamma_p = p$. Low-order moments, $p < 1$, are determined by a central scaling region and follow the diffusive law. The higher moments for the zonal displacements continue to follow the diffusive law, while for the meridional displacements are subdiffusive.

Conclusions

An outstanding problem in large-scale ocean dynamics is the characterization and representation of tracer and particle transport by mesoscale geostrophic eddies. In coarse resolution models used in climate studies the mesoscale turbulent transport is parameterized with a diffusion equation and an enhanced eddy diffusivity D. Taylor [1921] first showed that this approach is formally valid only for times longer than the decorrelation time of the turbulent eddies. Recent analysis of float data [Rupolo et al., 1996; Veneziani et al., 2004] and numerical simulations [Berloff and McWilliams, 2002; Reynolds, 2002b] found that the diffusive limit is either achieved after transients of more than one year or it is not reached at all because particles drift to regions with different statistics before asymptoting the diffusive limit. The time necessary to reach the diffusive limit is so long as a result of long memory introduced in the flow by long-lived coherent structures such as mesoscale vortices, meanders of swift currents, and planetary waves. In this paper we discussed techniques to characterize and parameterize dispersion at times shorter than the decorrelation time of the eddy velocity. First we illustrated the approach using a non-Markovian extension of the telegraph model. Then we applied the technique to ocean data from the Western North Atlantic.

The dispersive power of velocity field is typically defined in terms of the growth rate in time of the second moment of particle displacements $\langle x^2\rangle$. Dispersion is said to be normal if $\langle x^2\rangle \sim t$. Rupolo et al. [1996], Berloff and McWilliams [2002], and Reynolds [2002b] find that for periods between a few days and an year ocean dispersion is anomalous, i.e. $\langle x^2\rangle \sim t^{\gamma_2}$ with $\gamma_2 \neq 1$. Based on these results, a few studies have proposed mathematical models that reproduce the anomalous scaling of the second moment [e.g. Berloff and McWilliams, 2002; Reynolds, 2002a]. In this paper we showed that the behavior of the second moment is not a good indicator of dispersion in the anomalous regime. A better characterization of the dispersion process is given by the scaling of the general moments $\langle|x|^p\rangle$

at $t \to \infty$. We showed examples of stochastic processes and ocean float trajectories where $\langle |x|^p \rangle \sim t^{\gamma_p}$ at large times. The exponent γ_p is an important descriptor of the dispersive process. The small-p moments contain information about the behavior of most of the particles. The large-p moments are determined by relatively few tail-particles which have experienced large displacements. Depending on the details of the dispersive process, the much studied exponent γ_2 might lie on either of the two branches.

We have emphasized that γ_p provides more information than γ_2. But γ_p also contains less information than the full particle concentration. So what are the advantages of using γ_p? It has been our consistent experience that clean functional forms for γ_p emerge at relatively early times and with a modest number of particles. By contrast, convincingly demonstrating self-similar collapse of the concentration profiles is much more difficult.

The importance of identifying the central scaling region containing most of the particles cannot be overemphasized. Accurate parameterizations of the dispersive process must capture the behavior of the majority of the tracer particles and these lie in the central scaling region. The common practice in theoretical and observational studies of ocean dispersion is to focus exclusively on the behavior of the second moment. This approach does not provide sufficient information to identify the central scaling region. Mathematical models tuned to reproduce the scaling of the second moment will have poor skill in reproducing the behavior of most particles whenever diffusion is weakly self-similar.

We close with some remarks on the importance of our results for deriving mathematical models of dispersion. We have shown a few examples of dispersive processes for which $\gamma_p = p/\nu$ for small moments representing the central scaling region of the particle concentration. Zaslavsky [2002] shows that when $\nu = 2$ the dispersive process is well described by the diffusion equation. When $\nu \neq 2$ the dispersive process is described by Fractional-Fokker-Planck-Kolmogorov (FFPK) equations which are integro-differential equation with algebraically decaying kernels. The long-time behavior of the propagator core is the similarity solution (9) of the FFPK equation. Hence FFPK equations might be a useful model to characterize ocean dispersion on annual timescales.

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