Residual Circulation in the Ocean

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Abstract. The Transformed Eulerian Mean formalism greatly simplifies the study of eddy mean flow interactions. However the standard formalism suffers of a number of limitations that make it not suitable for typical oceanic conditions. Here we show how to extend the formalism to represent properly boundary conditions and regions with steep isopycnal slopes. This formalism is then used to derive a parameterization for mesoscale motions in the ocean.

Introduction

The ocean circulation is turbulent in the sense that motions on all spatial and temporal scales continuously interact. Thus, it is not possible to describe the large-scale, low-frequency circulation in isolation of the small and fast eddy motions. However ocean models used for climate studies cannot afford to explicitly resolve all scales of motion, given the limitations of today's computers, and must resort to parameterizing the effect of small-scale, high-frequency motions on the larger scales. In present climate models, the ocean horizontal grid resolution is O(100) km or larger. At this resolution, the mesoscale dynamics are sub-grid scale and their effects must be mostly parameterized. More powerful computers may decrease these scales to a marginal eddy resolution of O(25) km in the next ten years, but horizontal grids smaller than O(10) km are needed to adequately resolve the fluxes produced by mesoscale eddies (*Smith et al.*, 2000): even marginal eddy resolution requires some eddy parameterization.

Most climate models have adopted the Gent and McWilliams (hereafter GM) parameterization to represent unresolved mesoscale eddy motions. The key idea behind this parameterization is that mesoscale eddies adiabatically rearrange fluid parcels along isopycnals, without changing the density of individual water parcels (*Gent and McWilliams*, 1990). Consequently GM represents the eddy flux of a passive tracer through both a diffusion along isopycnals and an advection by an *eddy induced velocity*. These adiabatic conservation properties have led to a series of dramatic improvements in ocean models (*Danabasaoglu and McWilliams*, 1995).

Nevertheless there remain significant shortcomings of GM. Most notably, the scheme is not suited to deal with the ocean surface and lateral boundaries, and tends to overestimate mesoscale eddy fluxes in areas with large isopycnal slopes, which occur at frontal regions and in the surface mixed layer. Furthermore, the eddy-induced transport is chosen to mimic the effects of baroclinic instability, by extracting potential energy from the mean flow. In contrast a number of studies suggest that mesoscale eddies tend to homogenize potential vorticity (hereafter PV) along isopycnals (*Holland and Rhines*, 1980; *Rhines and Young*, 1995; *Wardle and Marshall*, 2000). Our goal is to derive a formalism in which the different choices can be compared and tested.

In this paper we consider how the Transformed Eulerian Mean (hereafter TEM) theory of *Andrews and McIntyre* (1976) can be used to improve our understanding of eddy motions in regions of steep isopycnal slopes and at the ocean boundaries. We then discuss the implications of this formalism for eddy parameterizations in climate models of the ocean.

Transformed Eulerian Mean and eddy parameterizations

The TEM theory of *Andrews and McIntyre* (1976) greatly simplifies the study of eddy mean flows interactions. The theory has also proved very useful to derive parameterizations of unresolved eddy motions (*Gent et al.*., 1995; *McDougall and McIntosh*, 1996; *Killworth*, 1997). However most progress in TEM theories has been restricted to zonal mean flows in the quasi-geostrophic (hereafter QG) approximation. This is clearly not satisfactory for oceanic applications, where mean flows are rarely zonal and quasi-geostrophic assumptions often fail. In this section, we review some basic results of TEM theory, and in the rest of the paper we will try to generalize the formalism to deal with the oceanic situation.

In conventional terms, the QG zonal mean problem takes the form,

$$\bar{u}_t - f\bar{v} = -\overline{u'v'} + \bar{\tau}, \quad (1)$$

$$f\bar{u}_z = -\bar{b}_y, \quad (2)$$

$$\bar{v}_y + \bar{w}_z = 0, \quad (3)$$

$$\bar{b}_t + \bar{w}\bar{b}_z = -\overline{v'b'} + \bar{S}. \quad (4)$$

Here (u, v, w) are the divergenceless velocity components in the (x, y, z) directions, $b = -g(\rho - \rho_0)/\rho_0$ is buoyancy, τ and S represent sources and sinks of momentum and buoyancy. Subscripts denote partial differentiation, overbars denote zonal average, and prime quantities are departures from the mean. The set (1) through (4), together with suitable boundary conditions, can in principle be solved for the mean state variables $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{b}$, given the mean nonconservative terms $\bar{\tau}$ and $\bar{S}$, and the two eddy flux terms.

The corresponding TEM equations hinge on the transformation $(\bar{v}, \bar{w}) \rightarrow (\bar{v}^\dagger, \bar{w}^\dagger)$, where the *residual circulation* is

$$(\bar{v}^\dagger, \bar{w}^\dagger) = (\bar{v}, \bar{w}) + \left(-\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right)\left(\frac{\overline{v'b'}}{\bar{b}_z}\right). \tag{5}$$

Then, under the conventional QG assumptions of small Rossby number and small isopycnal slope, the set (1) through (4) becomes

$$\bar{u}_t - f\bar{v}^\dagger = \nabla \cdot \boldsymbol{E} + \bar{\tau}, \tag{6}$$

$$f\bar{u}_z = -\bar{b}_y, \tag{7}$$

$$\bar{v}^\dagger_y + \bar{w}^\dagger_z = 0, \tag{8}$$

$$\bar{b}_t + \bar{w}^\dagger \bar{b}_z = \bar{S}, \tag{9}$$

where

$$\boldsymbol{E} = \left(-\overline{u'v'}, f\frac{\overline{v'b'}}{\bar{b}_z}\right), \tag{10}$$

is the Eliassen-Palm flux *Eliassen and Palm*, 1961; *Andrews and McIntyre*, 1976).

While the set (6) through (9) is equivalent to (1) through (4), the TEM has many well known advantages, both conceptual and practical. The first advantage is that the residual mean is related to the mass-weighted mean circulation in isopycnal coordinates. In practice, this circulation is simpler than the conventional Eulerian mean in regions where eddy fluxes are large. For example the thermally indirect Deacon cell in the Southern Ocean disappears in the residual circulation.

A second advantage is that, by regarding the residual circulation as being the mean circulation, the transformation (5) has removed the eddy fluxes from the buoyancy budget. As will be discussed below, this is a consequence of part of the eddy flux being advective in nature, and the remainder being negligible under quasi-geostrophic scaling. Notice, however, that in the ocean Reynolds' stresses are often negligible, and this advantage is not an issue, because there is only one eddy term $\overline{v'b'}$ in both the Eulerian and the transformed mean sets, the only difference being that the eddy flux term is moved from the buoyancy equation to the momentum equation.

Finally a third advantage is that the eddy forcing term of the transformed set in (6) is directly related to the eddy flux of QG potential vorticity, q,

$$\nabla \cdot \boldsymbol{E} = \overline{v'q'}. \tag{11}$$

Thus in the TEM formulation the interaction between mean and eddies can be expressed as the flux of a quasi-conserved tracer, the quasi-geostrophic PV. This result is extremely useful to derive eddy parameterizations, because, under most circumstances, eddy fluxes of quasi-conserved tracers can be related to the large scale local mean gradients. The idea is that eddy fluxes of quasi-conserved represent the deformation of mean tracer contours by small scale motions, and so are related to the local mean gradients (*Green*, 1970; *Plumb*, 1979;*Holland and Rhines*, 1980; *Rhines and Young*, 1982). This is not the case for momentum fluxes, because momentum can be radiated efficiently over long distances, defying any attempt to relate its fluxes to local mean gradients.

Countering these advantages are four disadvantages, all of which are relevant for the ocean. The first difficulty is that the assumption of small isopycnal slope on which QG theory relies, is rarely valid near the ocean surface, in frontal regions or in most of the Southern Ocean.

A second disadvantage is that diapycnal buoyancy fluxes drop away in the transformed budget. Diapycnal fluxes are second order in isopycnal slope and are negligible in QG theory. However, as the boundaries are approached, eddy fluxes develop a diabatic component and play an important role in the heat and salt budgets of the global ocean. The appearance of a diabatic eddy flux has a simple explanation: eddy motions are constrained to be parallel to the boundaries, because there is no mass flux across the boundaries, while the mean buoyancy surfaces can, and often do, intersect the boundaries (for example in response to atmospheric fluxes at the surface). *Plumb* (2003) shows the importance of the diabatic eddy flux to set the circulation in the two dimensional zonal-mean case. A goal of this paper is to derive a TEM theory in which diabatic eddy fluxes are not neglected.

A third difficulty occurs at the boundaries. Consider the upper boundary as an example. Equation (5) dictates that at the upper surface the residual vertical velocity is given by

$$\bar{w}^\dagger = \frac{\partial}{\partial y}\left(\frac{\overline{v'b'}}{\bar{b}_z}\right). \tag{12}$$

Thus $\bar{w}^\dagger$ is nonzero, whenever the horizontal buoyancy flux is nonzero (as it usually is). This is not a problem in theoretical calculations—one can apply this boundary condition as well as any other—but it does become an issue when eddies are to be parameterized and boundary conditions must be prescribed without a priori knowledge of the eddy field. The simplest way to deal with this nuisance, is to postulate an infinitesimally thin sheet (in a model the top layer) above the actual boundary, capped by an isothermal lid on which $\overline{v'b'} = 0$ and therefore $\bar{w}^\dagger = 0$ (*Bretherton*, 1996). Then the mass fluxes (12) into and out of the lower surface of the sheet are balanced by a finite horizontal mass flux within the sheet. While this flow might seem to be a mathematical construct, it is in fact a very real flow, occurring within the "surface zone" comprising isopycnals that outcrop at the mean surface, at certain times (*Held and Schhneider*, 1999). In the QG limit, this layer collapses to an infinitesimal thickness, but retains a finite mass flux, representing the return flow of the interior circulation. It is important to note that this flow must be parameterized in a model, if tracers are to be transported correctly.

Finally, a fourth difficulty is that the TEM formalism is best suited to deal with zonal flows, but ocean currents are strongly non-zonal because of topographic constraints. There have been previous attempts to extend the classical TEM formalism to deal with three-dimensional situations (*McDougall and McIntosh*, 1999; *Greatbatch*, 2001; *Plumb*, 1990), but these studies cannot be easily extended to address the three difficulties mentioned above.

These four difficulties cannot be easily addressed in general, but some progress can be made in particular cases. In what follows, we will extend the TEM formalism under the assumption that buoyancy eddy fluxes dominate over the momentum fluxes, as it is often the case in the ocean.

Residual Circulation and Residual Fluxes

The TEM formalism hinges on a transformation of the mean velocity field that captures most of the eddy flux. The procedure can be applied to any tracer equation. Consider a tracer of concentration c, that satisfies a conservation equation of the form,

$$c_t + \boldsymbol{u} \cdot \nabla c = S\{c\}, \tag{13}$$

where $S\{c\}$ represents sources and sinks of tracer. The velocity field is assumed to be divergenceless, i.e. $\nabla \cdot \boldsymbol{u} = 0$. For the present discussion, the tracer can be either active or passive. By active tracer, we mean a tracer that affects the momentum budget, like buoyancy, or potential vorticity. A tracer is instead considered passive if its concentration does not affect the velocity field $\boldsymbol{u}$, as for chemical compounds, or biological species.

Following a Reynolds decomposition of variables into a slowly changing mean and fluctuations, the mean conservation budget for the tracer c is

$$\bar{c}_t + \bar{\boldsymbol{u}} \cdot \nabla \bar{c} = -\nabla \cdot \boldsymbol{F}\{c\} + \bar{S}\{c\}, \tag{14}$$

where $\boldsymbol{F}\{c\} = \overline{\boldsymbol{u}'c'}$ is the eddy flux of c. The manipulation of such equations to obtain a transformed set hinges on the introduction of a nondivergent *residual circulation* $\bar{\boldsymbol{u}}^\dagger \equiv \bar{\boldsymbol{u}} + \nabla \times \boldsymbol{\Psi}$, where the choice of the vector streamfunction $\boldsymbol{\Psi}$ is left for the moment open. In terms of the residual circulation the budget in (14) becomes

$$\bar{c}_t + \bar{\boldsymbol{u}}^\dagger \cdot \nabla \bar{c} = -\nabla \cdot \boldsymbol{F}^\dagger\{c\} + \bar{S}\{c\}, \tag{15}$$

where we introduced the *residual flux*,

$$\boldsymbol{F}^\dagger\{c\} \equiv \boldsymbol{F}\{c\} - \boldsymbol{\Psi} \times \nabla \bar{c}. \tag{16}$$

We are free to drop the term $\nabla \times (\boldsymbol{\Psi}\bar{c})$ in the definition of $\boldsymbol{F}^\dagger\{c\}$, because this term is non-divergent and does not affect the evolution of $\bar{c}$.

The residual flux and the full flux have the same projection along the mean tracer gradient,

$$\boldsymbol{F}^\dagger\{c\} \cdot \nabla \bar{c} = \boldsymbol{F}\{c\} \cdot \nabla \bar{c}, \tag{17}$$

but have different projections in the plane orthogonal to $\nabla \bar{c}$. Thus the vector streamfunction $\boldsymbol{\Psi}$ can be chosen in such a way as to eliminate the flux component in the plane orthogonal to $\nabla \bar{c}$, the so-called *skew flux*, leaving just the flux component along $\nabla \bar{c}$ to remain in $\boldsymbol{F}^\dagger\{c\}$. This manipulation has advantages from a modeling perspective, because the parameterization problem is simpler for the down-gradient component of a flux, than for the skew component. We return to this point below.

The difference between $\boldsymbol{F}\{c\}$ and $\boldsymbol{F}^\dagger\{c\}$ is best illustrated if the eddy fluxes are related to the local mean gradient $\nabla \bar{c}$, as in

$$\boldsymbol{F}\{c\} = -K \cdot \nabla \bar{c}, \tag{18}$$

where K is a nine-component diffusivity tensor, and the dot product must be interpreted as a matrix product, i.e. $F_i\{c\} = -K_{ij}\partial_j \bar{c}$. The relationship in (18) might not seem to achieve much, because one is folding the complexities of the eddy structure from $\boldsymbol{F}\{c\}$ into K. However, by writing the eddy flux in this form, it becomes transparent that eddies can have advective-like and diffusive-like effects on the evolution of the mean tracer $\bar{c}$. Consider splitting the full tensor K into symmetric and antisymmetric components, K^s and K^a respectively. The symmetric part can be diagonalized and is likely to represent down-gradient diffusion parallel to the principal axes of the tensor. The antisymmetric part has an associated *skew flux* $\boldsymbol{F}^a$ given by

$$\boldsymbol{F}^a = -K^a \cdot \nabla\bar{c} = \boldsymbol{\Phi} \times \nabla\bar{c}, \tag{19}$$

where $\boldsymbol{\Phi} = -(K^a_{23}, K^a_{31}, K^a_{12})$. Thus the skew flux represents advection by a vector streamfunction $\boldsymbol{\Phi}$, rather than diffusion. This is confirmed by inspection of (19): the skew flux is perpendicular to $\nabla\bar{c}$ and does not dissipate mean tracer variance, because dissipation can only be achieved by fluxes crossing mean contours of $\bar{c}$. If we write the eddy and residual fluxes in terms of the diffusive and skew fluxes, we obtain

$$\boldsymbol{F}\{c\} = -K^s \cdot \nabla\bar{c} + \boldsymbol{\Phi} \times \nabla\bar{c}, \tag{20}$$

$$\boldsymbol{F}^\dagger\{c\} = -K^s \cdot \nabla\bar{c} + (\boldsymbol{\Phi} - \boldsymbol{\Psi}) \times \nabla\bar{c}. \tag{21}$$

It is evident that the choice $\boldsymbol{\Psi} = \boldsymbol{\Phi}$ eliminates the full skew component from $\boldsymbol{F}^\dagger\{c\}$.

Plumb (1979) argues that, for quasiconserved tracers, the eddy diffusivity (or, at least, its antisymmetric components) is likely to be the same for any quantity c. This is justifiable if nonconservative processes are negligible on the eddy turnover time scale, in which case K is purely determined by the kinematics of the eddy motions. Moreover, while the symmetric part of K is directly related to nonconservation and dispersion processes, the antisymmetric part is less dependent on these, in fact usually being nonzero even when such effects are negligible. For example in the case of small amplitude eddies, the antisymmetric part represents the difference between Lagrangian and Eulerian mean flows, the so-called Stokes drift. Thus the definition of $\boldsymbol{\Psi}$ is likely to be rather insensitive to what c actually is.

The advantage of introducing a residual circulation stems from the evidence that the skew component tends to dominate the eddy flux for waves with a large rotational component. Thus by introducing the residual circulation, we eliminate a large component from $\boldsymbol{F}\{c\}$ and we simplify the mean tracer budget. From a modeling perspective this simplifies the parameterization problem, because the residual flux is related to a diffusive transport and can be parameterized by appealing to mixing length arguments.

Transformed buoyancy budget

The standard formalism introduced in the previous section is purely kinematic, but it proves very useful to understand the physical properties of eddy fluxes. In the ocean interior, for example, motions across isopycnals surfaces are suppressed by the strong stratification. Thus K^s describes large mixing rates along mean isopycnal surfaces and much smaller diapycnal mixing rates, whereas K^a represents a mean circulation induced by along-isopycnal eddy motions. Consistent with this picture, one expects eddy fluxes of buoyancy to be dominated by the skew component, and eddy fluxes of other tracers to have an additional diffusive component that mixes along isopycnal surfaces. In this section we use the buoyancy budget to gain insight into the skew component.

The mean buoyancy budget is

$$\bar{b}_t + \bar{\boldsymbol{u}} \cdot \nabla\bar{b} = -\nabla \cdot \boldsymbol{F}\{b\} + \bar{S}\{b\}. \tag{22}$$

Following *Andrews and McIntyre* (1978, we separate the eddy buoyancy flux into components across and along the mean buoyancy gradient,

$$\boldsymbol{F}\{b\} = \frac{\overline{\boldsymbol{u}'b'} \cdot \nabla\bar{b}}{|\nabla\bar{b}|^2}\nabla\bar{b} - \frac{\overline{\boldsymbol{u}'b'} \times \nabla\bar{b}}{|\nabla\bar{b}|^2} \times \nabla\bar{b}. \tag{23}$$

The cross-gradient component is equivalent to a mean buoyancy advection, since

$$\nabla \cdot \left[\frac{\overline{\boldsymbol{u}'b'} \times \nabla\bar{b}}{|\nabla\bar{b}|^2} \times \nabla\bar{b}\right] = \left[\nabla \times \frac{\overline{\boldsymbol{u}'b'} \times \nabla\bar{b}}{|\nabla\bar{b}|^2}\right] \cdot \nabla\bar{b}. \tag{24}$$

This term is the skew flux discussed in the previous section. The choice of $\boldsymbol{\Psi}$ that removes the full skew component from the residual flux is then

$$\boldsymbol{\Psi} = -\frac{\overline{\boldsymbol{u}'b'} \times \nabla\bar{b}}{|\nabla\bar{b}|^2}. \tag{25}$$

With this definition the residual circulation and residual buoyancy flux are

$$\bar{\mathbf{u}}^\dagger = \bar{\boldsymbol{u}} + \nabla \times \boldsymbol{\Psi}, \qquad \boldsymbol{F}^\dagger\{b\} = \frac{\overline{\boldsymbol{u}'b'} \cdot \nabla\bar{b}}{|\nabla\bar{b}|^2}\nabla\bar{b}. \tag{26}$$

Note that the residual budget does not change if one adds to $\boldsymbol{\Psi}$ a generic vector parallel to $\nabla\bar{b}$. This gauge invariance will be used later on.

In the ocean interior, where diabatic processes are weak, the residual flux is small and the transformed buoyancy budget is to a first approximation, purely advective,

$$\bar{b}_t + \bar{\mathbf{u}}^\dagger \cdot \nabla\bar{b} = \bar{S}\{b\}. \tag{27}$$

This equation resembles a mean Lagrangian budget (there are no explicit eddy flux terms), and it suggests that in the ocean interior the residual circulation might resemble mean Lagrangian circulation. However as the boundaries are approached, eddy fluxes develop a diabatic component and the residual flux cannot be neglected. Thus in the "diabatic surface layer" the residual circulation differs from the the Lagrangian mean circulation.

The definitions in (26) are a possible choice of residual circulation. However, the transformed buoyancy budget retains the same general form under a more general definition of $\boldsymbol{\Psi}$ (*Treguier et al.*., 1997; *Held and Schneider*, 1999). Define a generalized residual streamfunction as

$$\boldsymbol{\Psi} \equiv -\frac{\overline{\boldsymbol{u}'b'} \times \nabla\bar{b}}{|\nabla\bar{b}|^2} - \frac{\overline{\boldsymbol{u}'b'} \cdot \nabla\bar{b}}{|\nabla\bar{b}|^2}\,\boldsymbol{\alpha}, \tag{28}$$

where $\boldsymbol{\alpha}$ is an arbitrary three dimensional vector, which may be a function of space. With this substitution, the residual budget (22) takes the form,

$$\bar{b}_t + \bar{\mathbf{u}}^\dagger \cdot \nabla\bar{b} = -\nabla \cdot \mathbf{F}^\dagger\{b\} + \bar{S}\{b\}, \tag{29}$$

where the residual buoyancy flux is

$$\mathbf{F}^\dagger\{b\} \equiv \frac{\overline{\boldsymbol{u}'b'} \cdot \nabla\bar{b}}{|\nabla\bar{b}|^2}\,\left[\nabla\bar{b} + \boldsymbol{\alpha} \times \nabla\bar{b}\right]. \tag{30}$$

The residual flux remains proportional to the diapycnal flux in this generalized formulation, but it points in a direction that depends on the choice of $\boldsymbol{\alpha}$, and not down the mean buoyancy gradient. Three choices of $\boldsymbol{\alpha}$ deserve special attention.

Isopycnal definition of the residual circulation

Setting $\boldsymbol{\alpha} \equiv \mathbf{0}$ in (28) provides the cleanest separation of buoyancy fluxes along and across mean isopycnals. The transformation to the residual circulation involves the along-mean isopycnal flux alone, as shown in (25), and the residual flux is just the diapycnal eddy flux, directed along the mean buoyancy gradient, as shown in (26). This is a natural choice in the ocean, because motions are largely adiabatic in the interior and isopycnals provide a very useful reference system (*McDougall and McIntosh*, 1996). However, residual mean theory was developed to study large scale flows in the atmosphere, where diabatic processes can be large and isopycnal slopes are close to horizontal, so that mean isopycnal coordinates are not as useful. As a result (25) and (26) are not used in the residual mean theory literature.

In Fig. 1a we draw a schematic of the residual circulation for an example in which the mean operator is a zonal average. In this case the vector streamfunction and the residual flux take the form,

$$\boldsymbol{\Psi} \;=\; -\frac{\overline{v'b'}\;\bar{b}_z - \overline{w'b'}\;\bar{b}_y}{\bar{b}_y^2 + \bar{b}_z^2}\;\hat{\boldsymbol{x}}, \tag{31}$$

$$\boldsymbol{F}^\dagger \;=\; \frac{\overline{v'b'}\;\bar{b}_y + \overline{w'b'}\;\bar{b}_z}{\bar{b}_y^2 + \bar{b}_z^2}\;\nabla\bar{b}. \tag{32}$$

The black lines represent the zonally averaged buoyancy contours that outcrop in response to some diabatic forcing at the surface. The blue lines show the circulation due to the eddy fluxes (the vector streamfunction), and the red arrow represents the downgradient residual flux. In the full three-dimensional definition, there is a horizontal recirculation in addition to the overturning circulation shown in the figure.

Notice that there is a flow into and out of the upper boundary, whenever the horizontal buoyancy flux is not zero at the surface, as in the case involving baroclinic eddies. The form of the vector streamfunction in (25) does not automatically guarantee that $w^\dagger = 0$ at the boundary. The vertical residual velocity at the surface is given by

$$\bar{w}^\dagger \;=\; \hat{\boldsymbol{z}} \cdot (\nabla \times \boldsymbol{\Psi}) \tag{33}$$

$$\;=\; \nabla \cdot \left(\frac{\bar{b}_z}{|\nabla\bar{b}|^2}\,\overline{\boldsymbol{u}'_H b'}\right), \tag{34}$$

where $\hat{\boldsymbol{z}}$ is the unit vector in the vertical direction, and $\overline{\boldsymbol{u}'_H b'}$ the horizontal buoyancy flux. Only in the presence of a mixed layer, where $\bar{b}_z \to 0$, does the flow through the surface vanish. In the absence of a mixed layer one can impose an extra layer at the top of the numerical model, as thin as one likes, within which buoyancy is vertically mixed, as shown in Fig. 1b. Any return flow not already occurring within the diabatic surface layer would be located within this thin layer. Notice, however, that enhanced turbulence at the boundaries often takes care of the problem: isopycnals are then mixed normal to the boundary, and the return flow flows parallel to the surface/topography.

Quasi-geostrophic definition of the residual circulation

To recover the standard QG definition of residual mean theory, we set

$$\boldsymbol{\alpha} \equiv -\frac{\hat{\boldsymbol{z}} \times \nabla\bar{b}}{\bar{b}_z}, \tag{35}$$

in (28), i.e. we choose $\boldsymbol{\alpha}$ proportional to the isopycnal slope (but directed normal to the slope). By direct substitution into (28), we obtain

$$\boldsymbol{\Psi} = \frac{(\overline{\boldsymbol{u}'b'} \times \bar{b}) \cdot \hat{\boldsymbol{z}}}{|\nabla\bar{b}|^2}\frac{\nabla\bar{b}}{\bar{b}_z} + \frac{\hat{\boldsymbol{z}} \times \overline{\boldsymbol{u}'b'}}{\bar{b}_z}. \tag{36}$$

Taking advantage of the gauge invariance, we can drop the term parallel to $\nabla\bar{b}$ in the definition of $\boldsymbol{\Psi}$. We then

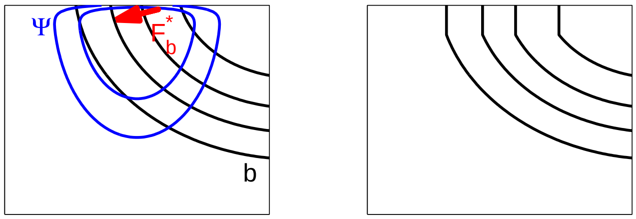

Figure 1. (a) Eddy-induced circulation (blue lines) and residual flux (red arrow) in a zonally averaged channel, according to the isopycnal definition in (31) and (32). The black lines are meant to represent mean buoyancy surfaces that outcrop in response to diabatic surface fluxes. The residual circulation does not vanish at the surface, except if there is a mixed layer on top where $\bar{b}_z \to 0$. (b) In the absence of a mixed layer one can always add an extra layer at the top of the model, as thin as one likes, within which buoyancy is vertically mixed. Any return flow not already occurring within the diabatic surface layer would be located within this thin layer. Notice that the residual diabatic flux becomes horizontal in the mixed layer and acts to redistribute laterally buoyancy anomalies generated by surface forcing.

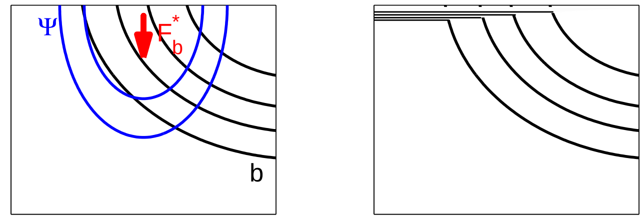

Figure 2. (a) Quasi-geostrophic definition of the residual circulation and residual flux for the same zonally averaged flow used for Fig. 1. In the QG definition, the residual circulation does not vanish at the surface, and the residual flux is directed in the vertical (lines have the same meaning as in Fig. 1). (b) The usual solution to deal with boundary conditions is to impose the existence of a thin layer at the top boundary capped by an isopycnal lid; then the return flow is concentrated in that thin layer.

recover the QG-based definition of residual circulation,

$$\mathbf{\Psi} = \frac{\hat{\boldsymbol{z}} \times \overline{\boldsymbol{u}'b'}}{\partial_z \bar{b}}, \tag{37}$$

and residual flux,

$$\mathbf{F}^\dagger = \frac{\overline{\boldsymbol{u}'b'} \cdot \nabla \bar{b}}{\partial_z \bar{b}}\ \hat{\boldsymbol{z}}. \tag{38}$$

The residual flux is ignored in the QG limit, because it is second order in isopycnal slope. However, at the boundaries QG scaling breaks down, and the residual flux cannot be ignored.

The residual circulation reduces to the definition given in (5), if the mean operator is interpreted as a zonal average,

$$\mathbf{\Psi} = -\frac{\overline{v'b'}}{\partial_z \bar{b}} \hat{\boldsymbol{x}}, \tag{39}$$

where $\hat{\boldsymbol{x}}$ is the unit vector along x. In Fig. 2a, we show the QG residual circulation and residual flux for the same configuration considered in Fig. 1a.

In the QG definition the residual vertical velocity is

$$\bar{w}^\dagger = \nabla \cdot \left(\frac{\overline{\boldsymbol{u}'_H b'}}{\bar{b}_z} \right); \tag{40}$$

thus there is a flow in and out of the boundary, whenever the horizontal buoyancy flux is nonzero at the surface, regardless of whether there is a mixed layer. The residual flux does not vanish either, if both the horizontal eddy buoyancy flux and the mean horizontal buoyancy gradient are nonzero at the surface. This flux however can only redistribute buoyancy in the vertical, because it has no horizontal component in the QG definition, so there is neither an advective nor a diffusive return flux within this layer, as shown in Fig. 2a. The usual solution to this is to impose the existence of a thin layer at the top boundary capped by an isopycnal lid; then $\bar{w}^\dagger = 0$ on that lid and the return flow is concentrated in that thin layer. But, as shown in Fig. 2b, the thin layer has a huge stability. This is of no consequence for theoretical considerations, but it is a serious problem if one wishes to use the residual formalism to parameterize the vertical structure of the return flow.

Notice that the isopycnal definitions (25) through (26) and the QG definition (37) through (38) are equivalent in the QG limit, because they differ by terms of second order in isopycnal slope. However in regions where isopycnals steepen, like at the ocean surface and at fronts, the two definitions are quite different.

Finally, *Plumb* (1990) shows that, in three dimensions, a more convenient definition of the QG vector streamfunction includes a term proportional to the eddy momentum fluxes. In this paper, we are not considering the effect of eddy momentum fluxes, and we will not pursue this point. However the approach of Plumb remains valid for any definition of $\mathbf{\Psi}$. The idea is to use the gauge invariance in the definition of $\mathbf{\Psi}$ to eliminate terms in the transformed momentum budget.

Held and Schneider definition of the residual circulation

Held and Schneider (1999) proposed an alternative definition of the residual circulation that satisfies no-normal flow boundary conditions at the surface for zonally averaged flows. This alternative definition can be extended to three dimensions by setting

$$\boldsymbol{\alpha} \equiv \frac{\hat{\boldsymbol{z}} \times \nabla \bar{b}}{|\nabla_H \bar{b}|^2} \partial_z \bar{b}, \tag{41}$$

in (28), i.e. $\boldsymbol{\alpha}$ is set proportional to the inverse slope, but directed normal to it. $\nabla_H b$ denotes the horizontal components of the buoyancy gradient. Taking advantage of the gauge invariance in the definition of $\boldsymbol{\Psi}$ and eliminating all terms directed along $\nabla \bar{b}$, we obtain

$$\boldsymbol{\Psi} = -\frac{\left(\overline{\boldsymbol{u}'b'} \times \nabla \bar{b}\right) \cdot \hat{\boldsymbol{z}}}{|\nabla_H \bar{b}|^2} \hat{\boldsymbol{z}} - \frac{\overline{w'b'}}{|\nabla_H \bar{b}|^2} \hat{\boldsymbol{z}} \times \nabla \bar{b}, \tag{42}$$

and the residual flux is

$$\mathbf{F}^\dagger = \frac{\overline{\boldsymbol{u}'b'} \cdot \nabla \bar{b}}{|\nabla_H \bar{b}|^2} \nabla_H \bar{b}. \tag{43}$$

The vector streamfunction is composed of two terms: the first term represents a circulation in the horizontal plane, the second term an overturning circulation. Vertical velocities are associated exclusively with the overturning component. Thus the residual circulation has no mass flux across the surface, because the overturning circulation is proportional to $\overline{w'b'}$ which vanishes at the surface (no-normal flow boundary conditions ensure that $w' = 0$ there). With this definition, the return flow is closed, regardless of whether there is a mixed layer at the ocean's surface (Fig. 3).

The advantage of having natural boundary conditions at the upper surface comes at a price. The Held and Schneider definition of residual circulation does not reduce to the QG one, because in regions of small isopycnal slope the vector $\boldsymbol{\alpha}$ defined in (41) is largest. In the zonally averaged problem, it can be shown that the full set of modified TEM equations that arises from this alternative definition loses some appealing properties: the forcing in the TEM equation is related to a vertical PV flux (*Plumb*, 2003), which has no obvious interpretation, and the definition is ill posed in regions with flat isopycnal surfaces.

The definition of residual circulation proposed by Held and Schneider can be generalized to deal with tilted boundaries. Define $\hat{\boldsymbol{n}}$ as the normal to the boundary and pick $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha} \equiv \frac{\hat{\boldsymbol{n}} \cdot \nabla \bar{b}}{|\nabla \bar{b}|^2 - |\hat{\boldsymbol{n}} \cdot \nabla \bar{b}|^2} \, \hat{\boldsymbol{n}} \times \nabla \bar{b}. \tag{44}$$

This choice gives the vector streamfunction,

$$\boldsymbol{\Psi} = -\frac{\overline{\boldsymbol{u}'b'} \cdot \hat{\boldsymbol{n}}}{|\nabla \bar{b}|^2 - |\hat{\boldsymbol{n}} \cdot \nabla \bar{b}|^2} \, \hat{\boldsymbol{n}} \times \nabla \bar{b} - \frac{\left(\overline{\boldsymbol{u}'b'} \times \nabla \bar{b}\right) \cdot \hat{\boldsymbol{n}}}{|\nabla \bar{b}|^2 - |\hat{\boldsymbol{n}} \cdot \nabla \bar{b}|^2} \, \hat{\boldsymbol{n}} \tag{45}$$

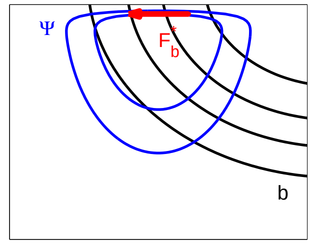

Figure 3. Held and Schneider definition of residual circulation and residual flux. Notation is explained in Fig. 1. The residual circulation vanishes naturally at the upper surface, regardless of whether there is a mixed layer. The residual flux is directed along the horizontal plane.

and the residual flux,

$$\mathbf{F}^\dagger = \frac{\overline{\boldsymbol{u}'b'} \cdot \nabla \bar{b}}{|\nabla \bar{b}|^2 - |\hat{\boldsymbol{n}} \cdot \nabla \bar{b}|^2} \, \left[\nabla \bar{b} - \left(\nabla \bar{b} \cdot \hat{\boldsymbol{n}}\right) \hat{\boldsymbol{n}}\right]. \tag{46}$$

This definition ensures that there are no mass flux across the tilted boundary, but suffers from the same limitations discussed for the choice of Held and Schneider.

Transformed potential vorticity budget

The transformed budget of buoyancy has been used to find appropriate definitions of the residual circulation. We now extend the analysis to tracers other than buoyancy to study the effect of the diffusive flux along isopycnal surfaces. This diffusive flux does not affect buoyancy because it acts along surfaces where buoyancy is homogeneous. We limit the discussion to potential vorticity, but the derivation is general and applies to any tracer.

The untransformed mean budget of the full Ertel PV, $P = g^{-1} \left(f\hat{\boldsymbol{z}} + \boldsymbol{\zeta}\right) \cdot \nabla b$, is

$$\bar{P}_t + \bar{\boldsymbol{u}} \cdot \nabla \bar{P} = -\nabla \cdot \boldsymbol{F}\{P\} + \bar{S}\{P\}, \tag{47}$$

where $\boldsymbol{\zeta} = \nabla \times \boldsymbol{u}$ is the relative vorticity, and

$$\boldsymbol{F}\{P\} = \overline{\boldsymbol{u}'P'} \tag{48}$$

The transformed budget can be written as

$$\bar{P}_t + \bar{\boldsymbol{u}}^\dagger \cdot \nabla \bar{P} = -\nabla \cdot \boldsymbol{F}^\dagger\{P\} + \bar{S}\{P\}, \tag{49}$$

where

$$\boldsymbol{F}^\dagger\{P\} \equiv \overline{\boldsymbol{u}'P'} - \boldsymbol{\Psi} \times \nabla \bar{P}, \tag{50}$$

is the residual PV flux. The form of the residual flux depends on the particular choice of $\boldsymbol{\Psi}$. For the choice based on an isopycnal formalism,

$$\boldsymbol{F}^\dagger\{P\} = \overline{\boldsymbol{u}'P'} + \frac{\overline{\boldsymbol{u}'b'} \times \nabla \bar{b}}{|\nabla \bar{b}|^2} \times \nabla \bar{P}. \tag{51}$$

In general, since we choose $\boldsymbol{\Psi}$ in order to remove the skew flux of buoyancy, we do not necessarily remove the skew flux of PV. However, it is often the case that skew components of the PV and buoyancy fluxes behave in the same way, because they are produced by the same advecting eddy velocity. After all, both tracers are quasi-conserved and their transport is dominated by eddy stirring, and not by irreversible mixing (which is likely to be different for the two). *Plumb* (2003) shows that this is indeed the case in a zonally averaged channel flow. Thus we expect that in general,

$$\frac{\overline{\boldsymbol{u}'b'} \times \nabla \bar{b}}{|\nabla \bar{b}|^2} = \frac{\overline{\boldsymbol{u}'P'} \times \nabla \bar{P}}{|\nabla \bar{P}|^2}, \tag{52}$$

except for a generic vector parallel to $\nabla \bar{b}$. But we already know that $\boldsymbol{\Psi}$ is defined apart from such a vector. We thus expect that

$$\boldsymbol{F}^{\dagger}\{P\} = \overline{\boldsymbol{u}'P'} + \frac{\overline{\boldsymbol{u}'P'} \times \nabla \bar{P}}{|\nabla \bar{P}|^2} \times \nabla \bar{P} \tag{53}$$

$$= \frac{\overline{\boldsymbol{u}'P'} \cdot \nabla \bar{P}}{|\nabla \bar{P}|^2} \nabla \bar{P}, \tag{54}$$

i.e. the residual flux is just the component of the PV flux down the mean PV gradient. This flux can be quite large, because it includes a large diffusive flux along isopycnal surfaces, the so-called Redi tensor (*Solomon* 1971).

Similar arguments can be repeated for all quasi-conserved tracers, like temperature and salinity. Here we focused on PV, because in the next section we use PV homogenization arguments together with TEM theory to derive a parameterization for mesoscale eddy fluxes.

Parameterizations

Residual mean theory provides a useful framework to derive eddy parameterizations, because it achieves a clean separation between the advective and diffusive properties of eddy fluxes. Here we show an example of how to use the residual PV budget, together with some physical insight, to derive a closure scheme for the eddy induced circulation.

We start by deriving a relationship between the residual circulation and eddy terms whose form will be, at least qualitatively, clear in many problems of interest. In a planetary geostrophic limit, i.e. for small *Ro*, but without any assumption of small isopycnal slope, nor of small amplitude eddies beyond what is implied by $Ro \ll 1$, the full PV flux, $\boldsymbol{F}\{P\}$, can be written as

$$\overline{\boldsymbol{u}'P'} = fg^{-1}\partial_z \overline{\boldsymbol{u}'b'} - g^{-1}\hat{\boldsymbol{z}} \times \nabla \frac{\overline{b'^2}}{2}. \tag{55}$$

Using the definition of residual flux in (16), and retaining only terms to first order in *Ro*, this relationship can also be written in terms of residual budgets,

$$\boldsymbol{F}^{\dagger}\{P\} = fg^{-1}\partial_z \boldsymbol{F}^{\dagger}\{b\} + fg^{-1}\partial_z \boldsymbol{\Psi} \times \nabla \bar{b} - g^{-1}\hat{\boldsymbol{z}} \times \nabla \frac{\overline{b'^2}}{2}. \tag{56}$$

The last term in the right hand side is nondivergent and plays no role in the residual PV budget. Thus it will be dropped in the following analysis.

Equation (56) is an important step toward a parameterization, because it provides a relationship between the eddy induced circulation, represented by the vector streamfunction $\boldsymbol{\Psi}$, and the residual fluxes of PV and buoyancy. As we have shown above, only the downgradient component of the full eddy flux remains in the residual, and its behavior is often well understood and can be parameterized. The more difficult along-gradient component, which unlike the down-gradient component is not easily related to eddy dissipation and stirring, does not appear in (56).

Rhines and Young (1982) have shown that the residual (downgradient) PV flux tends to homogenize PV along isopycnal surfaces. Thus we propose to parameterize the residual PV flux with a linear diffusive closure with a large diffusivity along isopycnal surfaces, and a smaller diapycnal component, i.e.,

$$\boldsymbol{F}^{\dagger}\{P\} = \kappa_I \frac{\nabla \bar{P} \times \nabla \bar{b}}{|\nabla \bar{b}|^2} \times \nabla \bar{b} - \kappa_D \frac{\nabla \bar{P} \cdot \nabla \bar{b}}{|\nabla \bar{b}|^2} \nabla \bar{b}, \tag{57}$$

with diffusivities $\kappa_I \gg \kappa_D$ to express the asymmetry between along and across isopycnal motions. This decomposition of the diffusive flux is referred to as the Redi tensor formalism (*Solomon*, 1971).

The residual (downgradient) buoyancy flux does not have any isopycnal component and can be expressed with a closure of the form,

$$\boldsymbol{F}^{\dagger}\{b\} = -\kappa_D \nabla \bar{b}. \tag{58}$$

Plugging the two closures (57) and (58) into (56) and projecting along isopycnal surfaces, we obtain

$$\partial_z \boldsymbol{\Psi} = (\kappa_I - \kappa_D) \frac{\nabla \bar{b}_z \times \nabla \bar{b}}{|\nabla \bar{b}|^2}, \tag{59}$$

where we neglected variations in f and we dropped an inconsequential nondivergent component. The vertical structure of the eddy diffusivities must be chosen so that $\boldsymbol{\Psi}$ satisfies no-normal flow boundary conditions.

In Fig. 4 we compare the PV homogenization closure in (59) with the GM closure. We consider a buoyancy front with no cross-frontal structure, so that the vector streamfunction has only one component, directed out of the page. For weak fronts the two closures are very similar (upper panels): in the limit of small isopycnal slope, the closure (59) does indeed reduce to GM,

$$\boldsymbol{\Psi} = (\kappa_I - \kappa_D)\partial_z \left(\frac{\nabla_H \bar{b} \times \hat{\boldsymbol{z}}}{\bar{b}_z} \right). \tag{60}$$

However, as the front steepens, the GM closure grows unbounded, while the PV homogenization closure remains well behaved. A second difference, not shown in the figure, is that for fronts with structure in both x and y, the PV homogenization closure, unlike GM, predicts an horizontal recirculation in addition to the overturning circulation.

The main point of this section was to show that it is possible to derive eddy closures based on the isopycnal definition of TEM. These closures have the advantage that they satisfy no-normal flow boundary conditions and remain valid for arbitrary isopycnal slopes.

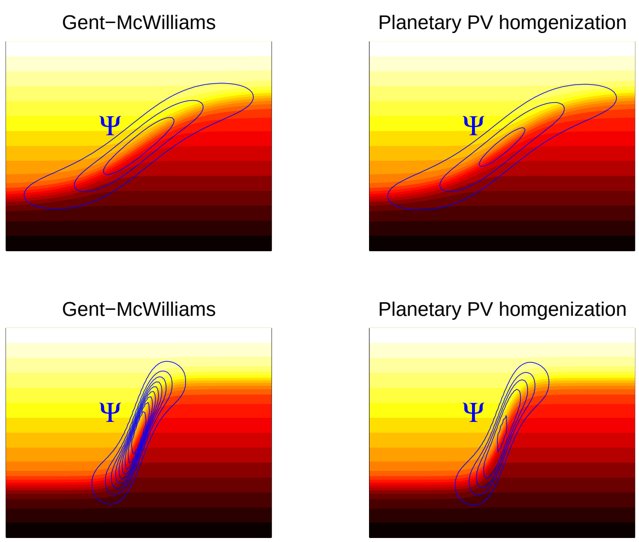

Figure 4. A comparison of the closure in (59), based on a PV homogenization argument, and the GM closure in (60), based on energetics arguments. As an example, we consider a two-dimensional buoyancy front. The blue contours show the eddy-induced circulation predicted by the two closure schemes. (For the PV closure we also had to impose the vanishing of $\boldsymbol{\Psi}$ at the boundaries). Both closures predict that eddies generate a clockwise overturning circulation that corresponds to a slumping of the front. However, the GM closure grows unbounded for steep fronts, while the PV based closure remains well behaved also for fronts with steep isopycnal slopes.

Conclusions

The Transformed Eulerian Mean theory was developed by *Andrews and McIntyre* (1976) to study eddy mean flow interactions in the atmosphere. The standard formalism suffers from four limitations that make it unsuitable for oceanic studies: 1) the theory assumes that isopycnal surfaces are close to horizontal, 2) diapycnal buoyancy fluxes are neglected, 3) the transformed velocity field does not satisfy no-normal flow boundary conditions, and 4) the theory is developed for zonal flows. In this paper we have shown that it is possible to overcome all four difficulties by modifying the definition of residual circulation. The basic idea is to split the eddy buoyancy flux into an along-isopycnal advective component (the eddy-induced velocity) and into a diapycnal component. This leads to a definition of a three-dimensional residual flow that describes the sum of the mean and the eddy-induced velocities. We referred to this definition as the Isopycnal Transformed Eulerian Mean.

We have also shown that the new formalism can be used to derive parameterizations for eddy fluxes in coarse resolution ocean models. The parameterization scheme we have derived is based on the principle that potential vorticity tends to be homogenized by eddies along isopycnal surfaces. The closure scheme satisfies no-normal flow boundary conditions and reduces to the widely used closure scheme of Gent and McWilliams in regions of weak isopycnal slope. The take-home message of this exercise is that the Isopycnal Transformed Eulerian Mean formalism provides a useful framework to develop eddy parameterizations that remain valid at the ocean boundaries and in strongly baroclinic zones.

In this paper we have limited the analysis to the buoyancy and potential vorticity budgets. We are currently extending these results to study the effect of eddy stresses in the momentum equation. Preliminary results suggest that, in the limit of small Rossby number, it is possible to obtain a closed set of equations for the residual circulation. That is, $\boldsymbol{\Psi}$ does not appear explicitly in the equations, but only as part of the residual circulation $\bar{\boldsymbol{u}}^{\dagger}$. Thus there is no need for a parameterization for the eddy-induced circulation, because the residual circulation becomes the only prognostic variable.

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